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ON THE DEVELOPMENT OF FUNCTIONS IN SERIES OF ORTHOGONAL POLYNOMIALS*

BY J. A. SHOHAT

Introduction. A function $p(x)$, called the *weight* or *characteristic function*, non-negative on a given interval (a, b), finite or infinite, and such that all *moments*†

$$\alpha_n = \int p(x)x^n dx, \qquad (n = 0, 1, 2, \cdots), \tag{1}$$

exist, with $\alpha_0 > 0$, gives rise, as is known [1],‡ to a system of orthogonal and normal Tchebycheff polynomials (*OP*)

$$\phi_n(x; p) \equiv \phi_n(x) = a_n x^n + \cdots, \quad (n = 0, 1, \cdots; a_n > 0), \tag{2}$$

uniquely determined by the following relations:

$$\int p(x)\phi_m(x)\phi_n(x)dx = \delta_{mn} = \begin{cases} 0, & m \neq n, \\ 1, & m = n, \end{cases} \tag{3}$$

$(m, n = 0, 1, \cdots)$. If $G_s(x) = \sum_{i=0}^{s} g_i x^i$ denotes an *arbitrary* polynomial of degree $\leqq s$ (subject in some cases to certain explicitly stated conditions), (2) yields

$$\int p(x)\phi_n(x)G_{n-1}(x)dx = 0, \qquad (n = 1, 2, \cdots). \tag{4}$$

The most important and best known *OP* are the so-called "classical" polynomials of

(J) Jacobi: (a, b) finite, say, $(-1, 1)$;
$$p(x) = (1 + x)^{\alpha-1}(1 - x)^{\beta-1}, \ (\alpha, \beta > 0).$$

(5) (L) Laguerre: $(a, b) = (0, \infty)$; $p(x) = x^{\alpha-1}e^{-x}$, $\quad (\alpha > 0)$.

(H) Hermite: $(a, b) = (-\infty, \infty)$; $p(x) = e^{-x^2}$.

We note the following important cases of (J):

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† Here and hereafter $\int$ means $\int_a^b$.

‡ Numbers in brackets refer to the Bibliography at the end.

Legendre polynomials ($\alpha=\beta=0$), trigonometric polynomials ($\alpha=\beta=1/2$).

The *OP* have many interesting and increasingly important applications in pure analysis, as well as in mathematical physics, mathematical statistics, and mechanics. We may mention approximate evaluation of definite integrals (mechanical quadratures), interpolation, curve fitting, certain oscillation problems in engineering mechanics, and investigation of certain classes of polynomials (for example, monotonic in (a, b)). The most important of such applications, which served to introduce the general *OP* in analysis, is their use in expanding *arbitrary* functions in series. In fact, making use of (3), we get the formal expansion

$$(6) \qquad f(x) \sim \sum_{n=0}^{\infty} f_n \phi_n(x), \qquad f_n = \int p(x) f(x) \phi_n(x) dx,$$

of any function $f(x)$, for which the integrals expressing the f_n exist. We see that (6) is built up in the same manner as the trigonometric expansion, the ordinary Fourier series. The latter employs the simplest orthogonal sequence $\{\sin nx, \cos nx\}$, $(n=0, 1, \cdots)$, while (6) employs orthogonal polynomials, which may be considered the next simplest orthogonal functions.

A fundamental question naturally arising in connection with the expansion (6) is that of its convergence. *Given the sequence* $\{\phi_n(x)\}$, *for what classes of functions* $f(x)$ *does* (6) *converge in* (a, b) *or in a part thereof?* The present paper is devoted to a discussion of various methods used in dealing with the convergence properties of the expansion (6). It is confined to a single real variable and to ordinary convergence. The proofs are in general but briefly sketched, if not omitted, our aim being to bring out general ideas. The topics treated can be classified as follows.

I. General properties of the expansion (6).

II. General methods for investigating the convergence of (6).

(a) Case of continuous functions; use of Weierstrass' theorem.

(b) Application of the theory of integral equations.

(c) Application of the general theory of orthogonal functions; Rademacher-Menchoff theorem, Lebesgue constants.

(d) The use of the asymptotic expression of $\phi_n(x)$, $(n\to\infty)$ (Dirichlet integral); equiconvergence.

III. Special methods for investigating the convergence of (6).

(*a*) From summability to convergence.

(*b*) Use of the closure equation.

(*c*) Use of the differential equation (for the classical *OP*).

(*d*) Comparison method.

These methods naturally vary in generality and power. This is best illustrated by applying the various methods to expansions in series of Legendre polynomials, the oldest and the best known *OP*. A comparison of the criteria of convergence thus obtained brings out the following fact: the more general the method, the less does it utilize the special character of the functions involved; hence, a gain in generality is accompanied, as a rule, by a loss in the preciseness of the results obtained.

The supreme goal, when dealing with (6), is to show that, regarding convergence, it behaves in a certain subinterval of (a, b) like the ordinary Fourier series expansion of $f(x)$ or of some function simply related to it. This constitutes what we call the *equiconvergence theorem*, of greatest importance in our theory. Indeed, we well know the wide range of validity of the Fourier series expansion, which makes it such a powerful analytical instrument. The classes of *OP* for which this goal is attainable are evidently the most interesting, and the method by which the goal has been attained is evidently the most powerful one.

I. Some General Properties of the Expansion (6).

1. *Various Representations of the Remainder in* (6). We rewrite (6) in the form

$$(7) \qquad f(x) = \sum_{i=0}^{n} f_i\phi_i(x) + R_n(x; f) \equiv S_n(x; f) + R_n(x; f),$$

whence, by use of (3) and (4), it follows that $R_n(x; G_n) \equiv 0$,

$$(8) \qquad \begin{aligned} 1 &= \int p(y) \sum_{i=0}^{n} \phi_i(x)\phi_i(y)dy \equiv \int p(y)K_n(x, y)dy, \\ S_n(x; f) &= \int p(y)f(y)K_n(x, y)dy, \end{aligned}$$

and, by use of Darboux's formula [2],

$$K_n(x, y) \equiv \sum_{i=0}^{n} \phi_i(x)\phi_i(y)$$

$$\equiv \frac{a_n}{a_{n+1}}\left[\frac{\phi_{n+1}(x)\phi_n(y) - \phi_n(x)\phi_{n+1}(y)}{x - y}\right],$$

(9)

$$K_n(x, x) = K_n(x) = \frac{a_n}{a_{n+1}}\left[\phi'_{n+1}(x)\phi_n(x) - \phi'_n(x)\phi_{n+1}(x)\right]$$

$$= \sum_{i=0}^{n} \phi_i^2(x).$$

$$S_n(x; f) = \frac{a_n}{a_{n+1}} \int p(y)f(y) \frac{\phi_{n+1}(x)\phi_n(y) - \phi_n(x)\phi_{n+1}(y)}{x - y} dy, \tag{10}$$

$$R_n(x) \equiv R_n(x; f) = \int p(y)[f(x) - f(y)]K_n(x, y)dy \tag{11}$$

$$= \frac{a_n}{a_{n+1}} \int p(y)[f(x) - f(y)] \frac{\phi_{n+1}(x)\phi_n(y) - \phi_n(x)\phi_{n+1}(y)}{x - y} dy,$$

$$R_n(x; f) = F(x) - \int p(y)[F(x) - F(y)]K_n(x, y)dy, \tag{12}$$

$$F(x) \equiv f(x) - G_n(x),$$

2. *Bessel Inequality. Closure.* The general assumption will be now made that $f(x)$ is of the class L_p^2, that is, $p(x)f^2(x)$ is L-integrable in (a, b). (Similarly, $f(x)$ being of the class L_p means $p(x)f(x)$ is L-integrable.) A ready application of (3) yields

$$\int p(y)\left[f(y) - \sum_{i=0}^{n} f_i\phi_i(y)\right]^2 dy = \int p(y)f^2(y)dy - \sum_{i=0}^{n} f_i^2,$$

which leads to the *Bessel inequality*:

$$\int p(y)[f(y) - S_n(y; f)]^2 dy = \int p(y)R_n^2(y)dy \tag{13}$$

$$= \int p(y)f^2(y)dy - \sum_{i=0}^{n} f_i^2 \geqq 0, \qquad (n = 0, 1, \cdots).$$

Hence we may write

$$\sum_{n=0}^{\infty} f_n^2 \text{ converges and is } \leqq \int p(y)f^2(y)dy. \tag{14}$$

If in (14) *we have the right to use the equality sign, then we have the so-called Parseval formula (closure equation)*, one of the most important in the theory of orthogonal functions:

$$\sum_{n=0}^{\infty} f_n^2 = \int p(y)f^2(y)dy, \tag{15}$$

where $f(x)$ is of class L_p^2, and $f_n = \int p(x)f(x)\phi_n(x)dx$. This, in connection with (13), leads to the equation

$$\int p(y)R_n^2(y)dy = \sum_{i=n+1}^{\infty} f_i^2 = o(1), \qquad (n \to \infty). \tag{16}$$

In the theory of *OP* it is shown: (i) *Parseval's formula always holds for* (a, b) *finite*; (ii) for (a, b) infinite its validity is intimately connected with the character, determined or indeterminate, of the *moment-problem* related to $p(x)$, that is, the problem whether the system of infinitely many equations

$$\int x^n d\psi(x) = \int p(x)x^n dx = \alpha_n, \qquad (n = 0, 1, \cdots),$$

where the unknown function $\psi(x)$ is monotonic non-decreasing in (a, b) with $\psi(a) = 0$ (and the left-hand integral is a Stieltjes integral), has or has not solutions distinct from the given one: $\psi(x) = \int_a^x p(x)dx$.* In particular, *Parseval's formula holds for the polynomials of Laguerre and Hermite. The validity of Parseval's formula is assumed throughout the subsequent discussion.*

3. *Consequences Derived from Parseval's Formula.* We know that

$$\lim_{n\to\infty} f_n = 0, \tag{17}$$

for any $f(x)$ of the class L_p^2. The set of relations

$$\int p(x)f(x)\phi_n(x)dx = 0, \qquad (n = 0, 1, \cdots), \tag{18}$$

implies $(p(x))^{1/2}f(x) = 0$ almost everywhere in (a, b).† In other words, two functions having the same *Fourier coefficients*

* The sharp distinction between the cases of (a, b) finite and infinite, as stated above, is clearly seen from the following example due to Stieltjes [3]:
$$\int_0^\infty e^{-x^{1/4}} \sin(x^{1/4})\, x^n dx = 0, \qquad (n = 0, 1, 2, \cdots).$$

† *If (a, b) is finite,* (18) *is valid for any $f(x)$ of the class L_p in (a, b).* In fact,

in (6) are equivalent, that is, their difference multiplied by $(p(x))^{1/2}$ vanishes almost everywhere in (a, b).*

The product

$$(p(x))^{1/2}S_n(x;f) \tag{19}$$

converges, for $n\to\infty$, *on the average* to $(p(x))^{1/2}f(x)$ on (a, b) [4]. It follows that a subsequence $(p(x))^{1/2}S_{n_k}(x;f)$ can be extracted which converges to $(p(x))^{1/2}f(x)$ in the ordinary sense $(n_k\to\infty)$ almost everywhere in (a, b). Hence, if $S_n(x;f)$ converges, for $n\to\infty$, on a set $E\subset(a, b)$ of positive measure (where we assume $p(x)\neq 0$), it necessarily converges to the value $f(x)$ almost everywhere on E.

We have also

$$\int p(x)f(x)F(x)dx = \sum_{n=0}^{\infty} f_nF_n, \tag{20}$$

where F is of class L_p^2, and

$$F_n=\int p(x)F(x)\phi_n(x)dx.$$

This is readily obtained by applying (15) to $f(x)\pm F(x)$. Assume further that

$$\int_c^d \frac{dx}{p(x)} \quad \text{exists}, \qquad (c, d)\subset(a, b). \tag{21}$$

Take in (20), $F(x)=1/p(x)$ in (c, d), and $F(x)=0$ elsewhere in (a, b); then $F_n=\int_c^d\phi_n(x)dx$, and we thus obtain the following important result:

$$\int_c^d f(x)dx = \sum_{n=0}^{\infty} f_n\int_c^d \phi_n(x)dx, \tag{22}$$

which tells us that *the expansion* (6), *whether convergent or not, can be integrated term by term in any interval* (c, d) *which is part*

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reduce (a, b) to $(-1, 1)$. Put $x=\cos\theta$ and denote $p(\cos\theta)f(\cos\theta)$ by $F(\theta)$. Then $\int_{-1}^{1}p(x)f(x)\phi_n(x)dx=0$, $(n=0, 1, \cdots)$, is equivalent to $\int_{-1}^{1}p(x)f(x)x^ndx=0$ or to $\int_0^{\pi}F(\theta)\cos n\theta d\theta=0$, $(n=0, 1, \cdots)$. Define $F_1(\theta)=F(\theta)$ in $(0, \pi)$, and $F_1(\theta)=-F(-\theta)$ in $(-\pi, 0)$. Then $\int_{-\pi}^{\pi}F_1(\theta)\begin{matrix}\cos\\ \sin\end{matrix} n\theta d\theta=0$ for all $n\geq 0$; hence, by the theory of trigonometric series, $F_1(\theta)=0$ almost everywhere in $(-\pi, \pi)$.

* Stieljes [3].

of (a, b), where (21) is satisfied, the convergence in (22) being uniform if d varies.*

We now turn again to the remainder $R_n(x; f)$ in (6) and assume, for definiteness, x to be a fixed point inside (a, b). Rewrite (11) as follows:

$$R_n(x; f) = \frac{a_n}{a_{n+1}}\left(\int_a^{x-\epsilon} + \int_{x-\epsilon}^{x+\epsilon} + \int_{x+\epsilon}^{b}\right) p(y)[f(x) - f(y)] \cdot K_n(x, y)dy \equiv i_1 + i_2 + i_3$$

$$= \frac{a_n}{a_{n+1}}\left\{\phi_{n+1}(x)\int_a^{x-\epsilon} p(y)\frac{f(x) - f(y)}{x - y}\phi_n(y)dy - \phi_n(x)\int_a^{x-\epsilon} p(y)\frac{f(x) - f(y)}{x - y}\phi_{n+1}(y)dy\right\} + i_2 + i_3. \tag{23}$$

(Here and hereafter ϵ denotes a sufficiently small positive quantity, properly chosen.) In i_1 (similar considerations apply to i_3) define

$$\psi(y) = \frac{f(x) - f(y)}{x - y} \text{ in } (a, x - \epsilon), \qquad \psi(y) = 0 \text{ in } (x + \epsilon, b).$$

Then

$$\int_a^{x-\epsilon} p(y)\frac{f(x) - f(y)}{x - y}\phi_n(y)dy = \int_a^b p(y)\psi(y)\phi_n(y)dy,$$

and we thus conclude, by (17), that

$$\lim_{n\to\infty}\int_a^{x-\epsilon} p(y)\frac{f(x) - f(y)}{x - y}\phi_n(y)dy = 0. \tag{24}$$

Furthermore, if (a, b) is finite, then a_n/a_{n+1} is bounded [1]: $a_n/a_{n+1} = O(1)$. If, in addition,

$$\phi_n(x) = O(1) \tag{25}$$

at the given point x, then

$$\lim_{n\to\infty}\frac{a_n}{a_{n+1}}\phi_{n+1}(x)\int_a^{x-\epsilon} p(y)\frac{f(x) - f(y)}{x - y}\phi_n(y)dy = 0. \tag{26}$$

It follows that the behavior of $R_n(x; f)$ depends here solely

* For Legendre polynomials (22) holds for any $f(x)$ of the class L [5].

upon its component i_2. The important conclusion resulting from the above considerations can be stated as follows. *For (a, b) finite, the convergence of the expansion* (6) *to $f(x)$ at a given point x where the sequence $\{\phi_n(x)\}$ remains bounded for $n\to\infty$ depends solely upon the nature of $f(x)$ in the immediate neighborhood of x* [6].

In other words, if two functions $f_1(x)$ and $f_2(x)$ coincide in an arbitrarily small neighborhood $(x-\epsilon, x+\epsilon)$ of the point x, then

$$\begin{aligned} &\lim\,[R_n(x; f_1) - R_n(x; f_2)] = 0, \\ &\lim\,[S_n(x; f_1) - S_n(x; f_2)] = 0, \end{aligned} \qquad (n \to \infty), \tag{27}$$

for $R_n(x; f_1-f_2)$ is here reduced to the components i_1 and i_3 in (23), written for $f_1(x)-f_2(x)$.

We now add the further condition that $f(x)$ *satisfies a Lipschitz condition* in the neighborhood of the point x, that is,

$$|f(y') - f(y'')| \leqq \lambda |y' - y''|, \tag{28}$$

$(x - \epsilon \leqq y', y'' \leqq x + \epsilon; \lambda = \text{const., independent of } y', y'')$.

Define $F(y)$ as follows:

$$F(y) = \frac{f(x) - f(y)}{x - y} \text{ in } (x - \epsilon, x + \epsilon),\ y \neq x,$$

$$= 0 \text{ elsewhere in } (a, b), \text{ and for } y = x.$$

Under condition (28), $F(y)$ is seen to be of the class L_p^2, and by the same reasoning as applied above to $i_{1,3}$, we show that here $\lim i_2 = 0$, so that $\lim R_n(x; f) = 0$, $(n\to\infty)$. The same conclusion holds if

$$|f(y') - f(y'')| \leqq \lambda |y' - y''|^{\alpha}, \text{ with } \alpha > 1/2, \tag{29}$$

where λ, y', y'' are as in (28) (*Lipschitz condition of order α*), *provided $p(x)$ is bounded in* $(x-\epsilon, x+\epsilon)$. Thus we have found, as an immediate consequence of Parseval's formula, that *the expansion* (6) *converges to $f(x)$ at any point x in the finite interval (a, b) in the neighborhood of which $f(x)$ satisfies a Lipschitz condition of order* 1 (*or of order $>1/2$, if $p(x)$ is bounded in this neighborhood*), *provided the sequence $\{\phi_n(x)\}$ is bounded at the point x under consideration.*

The latter condition is satisfied for various classes of *OP*, often uniformly in $(a+\epsilon, b-\epsilon)$, as illustrated by Jacobi poly-

nomials. (This is usually established by means of the asymptotic expression for $\phi_n(x)$.)

4. *Some Extremal Properties of the Expansion* (6). By means of (3) and (4) we readily derive the following relations:

$$\int p(y)\left[f(y) - \sum_{i=0}^{n}(f_i + h_i)\phi_i(y)\right]^2 dy = \int p(y)f^2(y) - \sum_{i=0}^{n} f_i^2 + \sum_{i=0}^{n} h_i^2,$$

where the h_i denote arbitrary constants; whence, by (13),

$$(30)\quad \int p(y)[f(y) - S_n(y;f)]^2 dy \leqq \int p(y)[f(y) - G_n(y)]^2 dy,$$

where the equality sign holds only for $G_n(y) \equiv S_n(y;f)$. This is an important extremal property which led Tchebycheff to the introduction of the general *OP* in analysis, *considering* (6) *as furnishing interpolation in the sense of least squares.*

II. General Methods for Investigating the Convergence Properties of the Expansion (6).

1. *Case of Continuous Functions.* The convergence of (6) evidently depends upon the order of magnitude (with respect to n) of the chosen polynomials $\phi_n(x)$ and of the coefficient f_n. While the former may be investigated once for all, by studying the properties of $p(x)$, the latter essentially depends on the nature of $f(x)$. We proceed to show that the study of the order of f_n is readily achieved in the important case where $f(x)$ *is continuous in the finite interval* (a, b). Here one naturally thinks first of Weierstrass' approximation theorem: *any function continuous in a given finite interval can be therein approximated uniformly and indefinitely by means of polynomials of ever increasing degrees.* Moreover, Tchebycheff has shown, that *among all polynomials of degree* $\leqq n$, *there exists a unique polynomial* $\Pi_n(x;f)$ *of "best approximation"* $(=E_n(f))$ *to* $f(x)$ *on* (a, b), that is,

$$(31)\quad E_n(f) = \max\,|f(x) - \Pi_n(x;f)| \leqq \max\,|f(x) - G_n(x)|,$$

$(a \leqq x \leqq b)$, where, by Weierstrass' theorem,

$$(32)\qquad \lim_{n\to\infty} E_n(f) = 0.$$

Here, as in many other problems dealing with continuous functions, Weierstrass' theorem proves a powerful and readily applicable tool.

(i) We have, first, by (4),

$$f_n = \int p(y)[f(x) - G_{n-1}(x)]dx$$

(33)

$$= \int p(x)[f(x) - \Pi_{n-1}(x; f)]\phi_n(x)dx.$$

Hence, applying Schwartz's inequality and making use of (31), we find that

$$|f_n\phi_n(x)| \leqq (\alpha_0)^{1/2}E_{n-1}(f)|\phi_n(x)|. \tag{34}$$

While this estimate, resulting from very simple considerations, is rather crude, it has the advantage of showing in many cases the convergence of (6) directly, without any further discussion. In fact, the order of magnitude of $E_n(f)$, depending upon the continuity properties of $f(x)$, is well known, thanks to the work of Lebesgue, de la Vallée-Poussin, S. Bernstein, and Dunham Jackson. Thus

$$|f^{(p)}(x') - f^{(p)}(x'')| \leqq \lambda|x' - x''|^{\alpha}, \quad (a \leqq x', x'' \leqq b; 0 < \alpha \leqq 1),$$

implies $E_n(f) = O(n^{-p-\alpha})$, $(p \geqq 0)$;

(35) $f^{(p)}(x)$ continuous in (a, b) implies $E_n(f) = o(n^{-p})$;

$|f(x+\delta) - f(x)| \cdot |\log|\delta||_{\delta \to 0} \to 0$ (Dini–Lipschitz condition)

implies $E_n(f) = o(1/\log n)$.

If now the order of $\phi_n(x)$ is known, say

$$|\phi_n(x)| = O(n^{\sigma}) \tag{36}$$

for a certain x, then, for the same x,

$$|f_n\phi_n(x)| = O(n^{\sigma}E_{n-1}(f)). \tag{37}$$

This, combined with (35), enables us to indicate at once classes of continuous functions for which (6) converges for the above x. If, for example, $\sigma = 0$, 1, then (6) certainly converges (absolutely) for the above values of x, if the first or the second

derivative of $f(x)$, respectively, satisfies in (a, b) a Lipschitz condition of an arbitrary positive order α, for then

$$|f_n\phi_n(x)| = O(n^{-1-\alpha}), \qquad (\alpha > 0).$$

The said convergence is uniform in any interval $\subset (a, b)$ in which $|\phi_n(x)| = O(n^\sigma)$, with $\sigma = 0, 1$.

(ii) More refined results are obtained from (12), with $G_n(x) \equiv \Pi_n(x; f)$. We thus get, applying Schwartz's inequality to the integral on the right, and using (3) once more,

$$(38)\quad |R_n(x; f)| \leqq E_n(f)\{1 + (K_n(x))^{1/2}\} = O(E_n(f)(K_n(x))^{1/2}).$$

$$(39)\quad |R_n(x; f)| = O(E_n(f)n^{\sigma+1/2}), \text{ if } |\phi_n(x)| = O(n^\sigma),$$

where $\sigma > -1$. We recall that $a_n/a_{n+1} = O(1)$ for (a, b) finite.

(iii) Still more refined results can be obtained [7, 8], if we again use (12), breaking up the integral on the right somewhat along the lines of (23):

$$\int_a^b = \int_a^{c+\epsilon} + \int_{c+\epsilon}^{x-\epsilon_n} + \int_{x-\epsilon_n}^{x+\epsilon_n} + \int_{x+\epsilon_n}^{x+\epsilon} + \int_{x+\epsilon}^{b}$$
$$\equiv i_1 + i_2 + i_3 + i_4 + i_5,$$
$$(a < c \leqq x \leqq d < b;\ \epsilon_n = o(1)(n \to \infty),\ |\phi_n(x)| = O(n^\sigma),\ \sigma > -1).$$

In $i_{1,5}$, where $|x-y| \geqq \epsilon$, we use Darboux's formula (9) for $K_n(x, y)$ and get integrals of the type $\int p(y)|\phi_n(y)|dy$ which $= O(1)$ (by Schwartz's inequality), so that

$$|i_{1,5}| = O(n^\sigma E_n(f)).$$

In $i_{2,4}$ put $x - y = u$; then

$$|i_{2,4}| = O(n^{2\sigma}E_n(f)) \int_{\epsilon_n}^{b-a} \frac{du}{u} = O(E_n(f) \cdot n^{2\sigma} |\log |\epsilon_n||).$$

Finally, in i_3 use $K_n(y) = O(n^{2\sigma+1})$, so that

$$|i_3| = O\left(E_n(f) \cdot n^{2\sigma+1} \int_{x-\epsilon_n}^{x+\epsilon_n} p(y)dy\right).$$

By taking $\epsilon_n = n^{-\beta}$, with a properly chosen $\beta > 0$, we can find the order of $R_n(x; f)$ and the class of continuous functions for which $\lim_{n\to\infty} R_n(x; f) = 0$.

NOTE. In the case under discussion, by (30) and (16),

$$(40)\quad \int p(y)R_n^2(y)dy \leqq \int p(y)\,|f(y)-\Pi_n(y;f)|^2 dy \leqq \alpha_0 E_n^2(f),$$

$$(41)\quad E_n(f) \geqq \frac{1}{\alpha_0^{1/2}}\left(\sum_{i=n+1}^{\infty} f_i^2\right)^{1/2}.$$

A lower bound for $E_n(f)$ is of importance in the theory of approximation. The relations (40) and (41), for a suitable orthogonal system $\{\phi_n(x)\}$, furnish such a lower bound, very acceptable for certain classes of functions [9].

2. *Application of the Theory of Integral Equations.* Rewrite (3) and (6), respectively, as

$$(42)\quad \int \Phi_m(x)\Phi_n(x)dx = \delta_{mn}, \qquad (m, n = 0, 1, \cdots),$$

$$(43)\quad F(x) \sim \sum_{n=0}^{\infty} f_n \Phi_n(x),$$

where

$$\Phi_n(x) = (p(x))^{1/2}\phi_n(x);\ F(x) = f(x)(p(x))^{1/2};\ f_n = \int F(x)\Phi_n(x)dx.$$

Consider the formal expression

$$(44)\quad K(x, y) \equiv \sum_{n=0}^{\infty} \frac{\Phi_n(x)\Phi_n(y)}{l_n} = (p(x)p(y))^{1/2}\sum_{n=0}^{\infty} \frac{\phi_n(x)\phi_n(y)}{l_n},$$

where l_n denote constants. For finite (a, b) the positive constants l_n in (44) may be chosen in infinitely many ways so that its right-hand member converges absolutely and uniformly in (a, b). Take, for example,

$$l_n = M_n^2 n^{1+\alpha},\ (\alpha > 0), \qquad M_n = \max\,|\phi_n(x)| \text{ in } (a, b).$$

The same holds for (a, b) infinite, if $(p(x))^{1/2}|\phi_n(x)|$ does not exceed in (a, b) a finite positive quantity M_n for each n (example: polynomials of Laguerre and Hermite). We have then, by virtue of orthogonality,

$$(45)\quad (p(x))^{1/2}\phi_n(x) = l_n \int (p(y))^{1/2}K(x, y)\phi_n(y)dy,$$

$$(46)\quad \Phi_n(x) = l_n \int K(x, y)\Phi_n(y)dy, \qquad (n = 0, 1, \cdots).$$

Thus, in cases under consideration, the functions $\{\Phi_n(x)\}$ are the fundamental functions for the linear integral equation (46), and the expansion (43), which is but a modification of (6), may be studied by applying the general theory of linear integral equations, more precisely, the Hilbert-Schmidt expansion theorem, which in its simplest form can be stated as follows. *Let $K(x, y)$ be a symmetric continuous kernel in the domain $a \leqq x, y \leqq b$, and let $\omega_n(x)$, $(n = 1, 2, \cdots)$, denote the corresponding fundamental functions. Any function of the form*

$$f(x) = \int K(x, y)h(y)dy, \tag{47}$$

where $h(x)$ is square-integrable in (a, b), can be expanded in a series according to the $\omega_n(x)$, which converges absolutely and uniformly; the expansion is of the form

$$f(x) = \sum_{n=1}^{\infty} \frac{h_n}{l_n} \omega_n(x), \tag{48}$$

where

$$h_n = \int h(y)\omega_n(y)dy,$$

and where l_n denote characteristic numbers.

However, we encounter here two serious difficulties. First, the actual construction of the kernel $K(x, y)$ in (45) and (46), depending largely upon the choice of the constants l_n in (44), is not so easy to carry out even in the simplest cases of Legendre and Hermite polynomials [10, 11, 12]; secondly, the criteria for convergence of (43) (or (6)) thus obtained are far too strict, for the method, due to its very generality, does not utilize sufficiently the individual properties of the orthogonal functions involved. Thus Weyl [11], applying the theory of integral equations to the expansion of a given function $f(x)$ according to Hermite functions $\{e^{-x^2/4}\phi_n(x)\}$ orthogonal in $(-\infty, \infty)$, finds the said expansion converges if $f(x)$ and $f'(x)$ are continuous and both integrals $\int_{-\infty}^{\infty} x^2 f^2(x)dx$, $\int_{-\infty}^{\infty} f'^2(x)dx$ exist. These criteria are far inferior to those obtained by other methods [13, 14]. Even more stringent are the criteria of W. Lebedeff [12] obtained by means of a different kernel. The same remark

applies to the discussion by the said writers of expansions in series of Laguerre polynomials.

A different application of the theory of integral equations to *OP* has been given by N. Kryloff [15]. His main point is the following generalized Schmidt's Lemma.

Consider a sequence $\{\omega_n(x)\}$, $(n=0, 1, \cdots)$, *of class* L^2, *orthogonal and normal in* (a, b), *for which Parseval's formula holds, that is, for any* $f(x)$ *of class* L^2,

$$\int f^2(x)dx = \sum_{n=0}^{\infty} f_n^2, \qquad f_n = \int f(x)\omega_n(x)dx.$$

If $\psi(x)$ *is of class* L^2 *and* $F(x, z)$ *is such that* $\int F^2(x, z)dz \leqq M$ (*independent of* z) *for any* z *in* (z_0, z_1), *then*

$$(49)\quad \int F(x, z)\psi(x)dx = \sum_{n=0}^{\infty} \int F(x, z)\omega_n(x)dx \cdot \int \psi(x)\omega_n(x)dx,$$

and the right-hand member converges absolutely and uniformly for $z_0 \leqq z \leqq z_1$. (See the Hilbert-Schmidt theorem given above.)

Suppose now we wish to investigate the convergence of the expansion

$$(6)\qquad f(x) = \sum_{n=0}^{\infty} f_n\phi_n(x), \qquad f_n = \int p(x)f(x)\phi_n(x)dx.$$

The ingenious idea of Kryloff is to *identify* (6) *with* (49), *and then to apply* the preceding Lemma. Choose, first, $F(x, z)$ and the sequence $\{\omega_n(x)\}$, then $\psi(x)$, so that

$$(50)\quad \begin{aligned} &\int F(x, z)\omega_n(x)dx = \phi_n(z), && (a \leqq z \leqq b), \\ &\int \psi(x)\omega_n(x)dx = \int p(x)f(x)\phi_n(x)dx, && (n = 0, 1, \cdots). \end{aligned}$$

If such a choice is possible, the right-hand member in (6) (where x is replaced by z) does become identical with that in (49), namely,

$$\sum_{n=0}^{\infty} f_n\phi_n(z) \equiv \sum_{n=0}^{\infty} \int \psi(x)\omega_n(x)dx \cdot \int F(x, z)\omega_n(x)dx.$$

Hence it converges absolutely and uniformly in (a, b) to the value of $f(x)$ (by virtue of (19)), if $F(x, z)$ satisfies the condition

of the Lemma. Kryloff applies this method to Jacobi polynomials in $(-1, 1)$ and shows, on the basis of their differential equation (see below, (85)), that such a choice as required by (50) is possible, if $\alpha, \beta > 1$, and *the expansion* (6) *converges absolutely and uniformly in* $(-1, 1)$, *if* $f(x)$ *and* $f'(x)$ *are of class* L_p^2.

3. *Application of the General Theory of Orthogonal Functions.* This is suggested by the very form of the expansion (43). The application in question, in order to attain the utmost in effectiveness, should be coupled with the special properties of the *OP* involved.

(i) *Use of Rademacher-Menchoff theorem* [16, 17]. This general theorem deals with a sequence $\{\omega_n(x)\}$ of functions orthogonal and normal in the finite interval (a, b), and tells us that *the expansion* $\sum_{n=0}^{\infty} c_n \omega_n(x)$ *converges almost everywhere in* (a, b), *if* $\sum_{n=2}^{\infty} c_n^2 \log^2 n$ *converges.** ($\omega_n(x)$ is of class L^2.)

This theorem, applied to (43), shows at once that (43) *converges to* $(p(x))^{1/2} f(x)$ *almost everywhere in the finite interval* (a, b), *if the coefficients* f_n *are such that* $\sum_{n=2}^{\infty} f_n^2 \log^2 n$ *converges.* The latter condition is satisfied in one of the following cases:

1. $0 < p(x) < M$, (M finite), or $0 < p(x) < M/((x-a)(b-x))^{1/2}$, $f(x)$ is of bounded variation in (a, b) [18].

2. $p(x) > 0$; $|f(x') - f(x'')| < \text{const.}/|\log|x' - x''||^{1+\epsilon}$, $(a \leqq x', x'' \leqq b)$† [19]. The Rademacher-Menchoff theorem lays the emphasis on the coefficients f_n in the expansion (6).

(ii) *Use of Lebesgue constants.* Here the emphasis is placed on the orthogonal functions employed. Using the expressions (11) and (12) for the remainder $R_n(x; f)$ in the expansion (6), we are led to introduce the Lebesgue constants:

$$\rho_n(x) = \int p(y) \left| K_n(x, y) \right| dy,$$

$$\bar{\rho}_n(x) = \int \left| \sum_{i=0}^{n} \Phi_i(x) \Phi_i(y) \right| dy, \tag{51}$$

* Note the presence of the factor $\log^2 n$. From the mere convergence of $\sum_{n=2}^{\infty} c_n^2$ follows only (by the Riesz-Fischer theorem) the existence of a function $f(x)$ of class L^2 such that $c_n = \int f(x) \omega_n(x) dx$, $(n = 0, 1, \cdots)$.

† And, a fortiori, if $f(x)$ satisfies in (a, b) a Lipschitz condition of an arbitrarily given order α (>0).

where x is given in (a, b), $(n=0, 1, \cdots)$, and where $\rho_n = \max \rho_n(x)$, $\bar{\rho}_n = \max \bar{\rho}_n(x)$ for $a \leqq x \leqq b$. The symbol max here and everywhere means, when necessary, *upper bound*. Hence, if $f(x)$ is bounded in (a, b),

$$(52) \quad |R_n(x; f)| \leqq 2M\rho_n(x) \leqq 2M\rho_n, \quad (a \leqq x \leqq b;\ |f(x)| \leqq M),$$

by (11); and if $f(x)$ is continuous in the finite interval (a, b),

$$(53) \quad |R_n(x; f)| \leqq E_n(f)\{1 + \rho_n(x)\} \leqq E_n(f)(1 + \rho_n),$$

$(a \leqq x \leqq b)$, by (12), where $G_n(x) \equiv \Pi_n(x; f)$. Here $\rho_n(x)$ is the upper bound of $S_n(x; f)$ for all $f(x)$ such that $|f(x)| \leqq 1$ in (a, b). In fact, for such $f(x)$,

$$|S_n(x; f)| \leqq \int p(y)\,|K_n(x, y)|\,dy = \rho_n(x).$$

Moreover, this upper bound is actually attained by the function $f(y) = \operatorname{sgn} K_n(x, y)$.* Equation (53), by virtue of (32), shows at once that *if* $\{\rho_n(x)\}$ *is bounded, then* (6) *converges to the value* $f(x)$ *at the given point* x *for any continuous* $f(x)$, the said convergence being uniform in any interval $\subset (a, b)$ where the Lebesgue constants remain uniformly bounded (in x and n). On the other hand, if the sequence $\{\rho_n(x)\}$ is unbounded, it can be shown, following Haar [20], that there exist continuous functions whose expansions (6) diverge at the point x. In any case, if the order, with respect to n, of $\rho_n(x)$ or ρ_n is known, equation (53), where (a, b) is assumed to be finite, will show at a glance for what classes of continuous functions (6) converges, by means of (35). If, for example, $\rho_n(x) = O(\log n)$, then (6) converges to $f(x)$ at the given point x, provided $f(x)$ satisfies in (a, b) a Dini-Lipschitz condition. In this connection it was shown by Rademacher [16] that

$$(54) \qquad \bar{\rho}_n(x) = O(n^{1/2}(\log n)^{3/2+\epsilon})$$

almost everywhere in (a, b); hence, (43) *converges to* $(p(x))^{1/2}f(x)$ *almost everywhere in* (a, b) *if* $f(x)$ *satisfies in* (a, b) *a Lipschitz condition of order* $> 1/2$.

* Note that $\operatorname{sgn} a = +1, -1, 0$, corresponding to $a > 0, < 0, = 0$.

The above discussion clearly shows the importance of the so-called "zero-" or "kernel series" (série-noyau, Kogbetliantz):*

$$(55) \qquad \sum_{n=0}^{\infty} \Phi_n(x)\Phi_n(y) \equiv (p(x)p(y))^{1/2} \sum_{n=0}^{\infty} \phi_n(x)\phi_n(y).$$

To illustrate, assume that

$$(56) \quad \sum_{n=0}^{\infty} \phi_n(x)\phi_n(y) = 0 \text{ for } x \neq y, \qquad (a \leqq c \leqq x,\ y \leqq d \leqq b),$$

uniformly if $|x-y| \geqq \epsilon$. Assume further, in order to simplify the discussion and to bring out more clearly the underlying ideas, that (56) holds in (a, b),† and let $f(x)$ be continuous at a certain interior point x (where $p(x) \neq 0$). We have then‡

$$S_n(x; f) = \left(\int_a^{x-\epsilon} + \int_{x-\epsilon}^{x+\epsilon} + \int_{x+\epsilon}^{b} \right) p(y)f(y) \sum_{i=0}^{n} \phi_i(x)\phi_i(y)dy$$
$$\equiv i_1 + i_2 + i_3,$$

where $i_1 = o(1)$, $i_3 = o(1)$, by (56);

$$i_2 = f(x) \int_{x-\epsilon}^{x+\epsilon} p(y) \sum_{i=0}^{n} \phi_i(x)\phi_i(y)dy$$
$$+ \int_{x-\epsilon}^{x+\epsilon} p(y)[f(y) - f(x)] \sum_{i=0}^{n} \phi_i(x)\phi_i(y)dy = i_2' + i_2'';$$

---

* A penetrating investigation of this and related zero-series by E. Kogbetliantz for various classes of *OP* has yielded in recent years extremely general results regarding convergence and summability of the expansion (6) (see, for example, [14]).

† Ordinarily (56) is satisfied when the series on the left is subjected to a certain process of summation, so that the conclusions drawn pertain not to ordinary convergence, but to summability. The latter, under additional hypotheses, may lead to ordinary convergence (see below, p. 72).

‡ The end points need special attention. Thus, if (a, b) is finite, we generally write $\int_a^{x-\epsilon} = \int_a^{a+\epsilon_n} + \int_{a+\epsilon_n}^{x-\epsilon}$, where $\epsilon_n = o(1)$ is properly chosen, and if (a, b) is infinite, say $a = -\infty$, then we write $\int_{-\infty}^{-G} + \int_{-G}^{x-\epsilon}$, where G (>0) is sufficiently large (or even depends on n and increases indefinitely with n). Similar remarks apply to b. In other words, we must properly specify the behavior of $f(x)$ near the end points.

$$i_2' = f(x)\left(\int_a^b - \int_a^{x-\epsilon} - \int_{x+\epsilon}^b\right) p(y) \sum_{i=0}^{n} \phi_i(x)\phi_i(y)dy$$

$$= f(x) + o(1) \qquad \text{(by (8) and (56))};$$

$$|\, i_2'' \,| \leqq \eta(\epsilon)\left(\int_a^b + \int_a^{x-\epsilon} + \int_{x+\epsilon}^b\right) p(y) \left| \sum_{i=0}^{n} \phi_i(x)\phi_i(y) \right| dy$$

$$= \eta(\epsilon)(\rho_n(x) + o(1));$$

$$(|\, f(y) - f(x) \,| \leqq \eta(\epsilon) \text{ for } x - \epsilon \leqq y \leqq x + \epsilon; \eta(\epsilon) \to 0, \text{ as } \epsilon \to 0).$$

Thus, we again come across the Lebesgue constants, namely,

$$(57)\quad S_n(x; f) = f(x) + \eta_1(\epsilon)\rho_n(x) + o(1), \quad (\eta_1(\epsilon) \to 0, \text{ as } \epsilon \to 0),$$

and this shows that $\lim_{n\to\infty} S_n(x; f) = f(x)$, *if the sequence* $\{\rho_n(x)\}$ *is bounded*, even though the continuity of $f(x)$ was here assumed at the point x only.* The important point here is that, due to (56),

$$(58)\quad \rho_n(x) = \int_{x-\epsilon}^{x+\epsilon} p(y) \left| \sum_{i=0}^{n} \phi_i(x)\phi_i(y) \right| dy + o(1), \qquad (n \to \infty).$$

In general, the estimate of $\rho_n(x)$ requires the use of the asymptotic expression for $\phi_n(x)$. Another method based on a theorem of Fubini will be omitted [21].

4. *Making Use of the Asymptotic Expression of* $\phi_n(x)$ $(n \to \infty)$. One can foresee that this method will prove the most powerful, for the said asymptotic expression is, we may say, a synthesis of the most intimate properties of $\phi_n(x)$. The essential features of this method may be exhibited as follows. For various classes of *OP* the asymptotic expression of $\phi_n(x)$, for $n \to \infty$, has in the first approximation the following form:

$$(59)\quad \phi_n(x) = A_n\left[\cos(n^q\psi + \omega_n) + \frac{B_n}{n^p}\right], \qquad (q, p > 0),$$

and this holds in a certain subinterval of (a, b), where A_n, ψ, ω_n, B_n are certain functions of x, with ω_n, A_n, B_n depending generally on n, and A_n, B_n remaining bounded for $n \to \infty$. Thus, for example, for Legendre polynomials [2, 22]:

* In case $f(x)$ is of bounded variation in the neighborhood of the given point x (the usual assumption made in dealing with (6)), the previous analysis, somewhat refined, would introduce the quantities $f(x \pm 0)$.

$$P_n(x) = \left(\frac{2}{2n+1}\right)^{1/2}\phi_n(x, 1) = \frac{4}{\pi}\cdot\frac{2\cdot 4\cdots 2n}{1\cdot 3\cdots(2n+1)}$$

$$\cdot\frac{1}{(2\sin\phi)^{1/2}}\cdot\left\{\cos\left[\left(n+\frac{1}{2}\right)\phi-\frac{\pi}{4}\right]+\frac{1}{2}\cdot\frac{1}{2n+3}\right.$$

$$\cdot\frac{\cos\left[\left(n+\frac{3}{2}\right)\phi-\frac{3\pi}{4}\right]}{2\sin\phi}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{1\cdot 3}{(2n+3)(2n+5)} \tag{60}$$

$$\left.\cdot\frac{\cos\left[\left(n+\frac{5}{2}\right)\phi-\frac{5\pi}{4}\right]}{(2\sin\phi)^2}+\cdots\right\}$$

$$=\left(\frac{2}{n\pi\sin\phi}\right)^{1/2}\left\{\cos\left[\left(n+\frac{1}{2}\right)\phi-\frac{\pi}{4}\right]+\frac{B_n(\phi)}{n}\right\},$$

$$(-1+\epsilon\leqq x=\cos\phi\leqq 1-\epsilon).$$

Similarly, for Hermite polynomials [13]:

$$\phi_n(x) = e^{x^2/2}\cdot\pi^{-1/2}\cdot\left(\frac{2}{n}\right)^{1/4}\cdot\left\{\cos\left(x(2n)^{1/2}+\frac{n\pi}{2}\right)\right.$$

$$\left.+\left(\frac{x^3}{6}-\frac{x}{2}\right)\cdot(2n)^{-1/2}\cdot\sin\left(x(2n)^{1/2}+\frac{n\pi}{2}\right)+\frac{B_n(x)}{n}\right\}, \tag{61}$$

where $|x|\leqq A$ is finite and arbitrarily fixed.

In the first place, we find, by means of Darboux's formulas (9), an asymptotic expression for $K_n(x, y)$ and $K_n(x)$, x being a certain given point belonging to the interval where (59) is valid. Next, again rewrite (10) in the form

$$S_n(x; f) = \left(\int_a^{x-\epsilon}+\int_{x-\epsilon}^{x+\epsilon}+\int_{x+\epsilon}^{b}\right)p(y)f(y)K_n(x, y)dy$$

$$\equiv i_1+i_2+i_3, \tag{62}$$

making the usual assumption that $f(x)$ is of bounded variation in the neighborhood of the point x. The most difficult part of the investigation, which requires great ingenuity (and often laborious computation) is to show that i_1 and i_3 tend to 0, as n tends to infinity, for the asymptotic expression (59) does not hold, as a rule, at the end points. Here we must specify the proper

behavior of $f(x)$. The integrals i_1 and i_3 having been disposed of, we proceed to the relatively simple investigation of i_2. Using (59) and making a proper change of the variable of integration, we are led to a finite number of integrals of the form

$$\int_{\alpha}^{\beta} u(z) \begin{matrix}\sin\\ \cos\end{matrix} mz\,dz, \qquad (\alpha, \beta \text{ finite or infinite}), \tag{63}$$

$$\int_{0}^{h} u(z) \frac{\sin mz}{\sin z}\,dz, \qquad \text{Dirichlet integral}, \tag{64}$$

$$\int_{h}^{k} u(z) \frac{\sin mz}{\sin z}\,dz, \qquad (0 < h < k \leqq \pi/2). \tag{65}$$

The behavior, for $m \to \infty$, of these integrals is well known from the theory of trigonometric series [4]. Thus, $(63) = o(1)$, if $|u(z)|$ is L-integrable in (α, β) (Riemann-Lebesgue theorem); $(64) \to (\pi/2)u(+0)$, if $u(z)$ is of bounded variation in $(0, \pi/2)$; $(65) \to o(1)$, for the same $u(z)$. These three limiting relations enable us to complete our investigation; (64) here furnishes the desired limit of $S_n(x; f)$, $(n \to \infty)$, in terms of $f(x \pm 0)$. The generality of the results obtained in this manner depends upon two factors: (i) the thoroughness of the study of the components $i_{1,3}$ in (62), by which we can avoid unnecessarily heavy restrictions to be imposed upon the differentiability and integrability properties of $f(x)$, also upon its behavior at the end points; (ii) the more or less comprehensive character of the asymptotic expression used, that is, its range of validity, order of magnitude, and nature of the remainder. Thus, Kogbetliantz [14] has obtained very general results on the convergence (and summability) of series according to Hermite or Laguerre polynomials, by using asymptotic expressions for $\phi_n(x)$ *valid over an interval which increases indefinitely with* n. Haar [23], making use of the second approximation in (59), was able to establish the most interesting result in the theory of Legendre polynomials—the so-called "equiconvergence theorem" (see Introduction): $f(x)$ *being of the class* L^2, *its expansion* (6) *in series of Legendre polynomials behaves, with regard to convergence or divergence at any interior point* $x = \cos\theta$, $(0 < \theta < \pi)$, *like the Fourier cosine-series expansion of* $f(\cos\theta)$, *that is, denoting by* $\sigma_n(\theta)$ *the partial sum of the latter series, we have*

$$\lim_{n\to\infty} [S_n(\cos\theta) - \sigma_n(\theta)] = 0, \qquad (0 < \theta < \pi).$$

Quite recently, Szegö [24] has developed for Jacobi polynomials with arbitrary $\alpha, \beta(>0)$ (see (5)) asymptotic expressions in *terms of Bessel Functions*, instead of trigonometric functions as in (59)–(61). His asymptotic expressions have the advantage of *being valid uniformly in any left-hand neighborhood of the end point* $x=1$. Thus one of the difficulties mentioned above is obviated, and we obtain in a simple manner the asymptotic expression of the Lebesgue constants and of many other definite integrals involving Jacobi polynomials. The most important application which Szegö makes of his asymptotic expressions is to derive a general "equiconvergence theorem," which, in case of Legendre polynomials, holds even for a more general class of functions than that in Haar's theorem (see below, p. 78).

In order to illustrate the difficulties arising in case of an infinite interval, we proceed to sketch the ingenious method of Uspensky in dealing with Hermite polynomials [13]. We write here

$$S_n(x; f) = \left(\int_{-\infty}^{-G} + \int_{-G}^{G} + \int_{G}^{\infty}\right) e^{-y^2} f(y) K_n(x, y)\, dy$$
$$\equiv i_1 + i_2 + i_3, \qquad (G>0, \text{ sufficiently large}).$$

The integral i_2 is reduced to the Dirichlet integral as usual, and furnishes in the limit, for $n\to\infty$, $[f(x+0)+f(x-0)]/2$, ($f(x)$ being of bounded variation in the neighborhood of the given point x). We now turn to i_3 (the analysis is similar for i_1). First, by Schwartz's inequality,

$$i_3^2 \leqq \int_G^\infty e^{-y^2} f^2(y)\, dy \cdot \int_G^\infty e^{-y^2} K_n^2(x, y)\, dy. \tag{66}$$

The first factor in (66) does not cause trouble under the liberal assumption

$$\int_A^\infty e^{-y^2} f^2(y)\, dy, \quad \text{and also} \quad \int_{-\infty}^{-A} e^{-y^2} f^2(y)\, dy, \tag{67}$$

exist for some $A > 0$.

We proceed to show that the second factor$\to 0$ as $n\to\infty$. By (9) and (10),

$$\phi'_{n+1}(x)\phi_n(x) - \phi'_n(x)\phi_{n+1}(x) = \frac{(2n+2)^{1/2}}{2}\int_{-\infty}^{\infty} e^{-y^2}\beta_n^2(y)dy$$

$$(68)\qquad \equiv I_n = \left(\frac{2n+2}{2}\right)^{1/2}\left(\int_{-\infty}^{-G} + \int_{-G}^{G} + \int_{G}^{\infty}\right)$$

$$\equiv I'_n + I''_n + I'''_n;$$

$$\beta_n(x,y) = \frac{\phi'_{n+1}(x)\phi_n(y) - \phi'_n(x)\phi_{n+1}(y)}{x-y},$$

where x is fixed and finite. The ingenious method of Uspensky consists in first evaluating asymptotically I_n, as a whole, using on the left side of (68) the asymptotic expressions of $\phi_n(x)$, $\phi'_n(x)$ (the latter also being a Hermite polynomial), then evaluating asymptotically I''_n, by using the above asymptotic expressions *under the integral sign* (the interval of integration being finite). *By subtraction we find at once the order of magnitude of* I'_n, I'''_n, and this completes the discussion of i_3, *without any further restriction being imposed on* $f(x)$. Laguerre polynomials are treated in a similar way. The criteria of convergence of (6) thus obtained are very sensitive indeed.

III. Some Special Methods for Investigating the Convergence Properties of the Expansion (6).

1. *From Summability to Convergence.* It is well known [25] that the Césaro (C), Abel (A), or Euler (E) summability of an infinite series $\sum_{n=0}^{\infty}u_n \equiv U$ implies its ordinary convergence if u_n is properly restricted. Thus, (i) if U is (C, k) summable, and $u_n = O(1/n)$, then the series converges (Hardy); (ii) if U is A-summable, and $u_n = O(1/n)$, then the series converges (Littlewood). The existence of generating functions for the classical OP, that is,

$$\Phi(x,t) = \sum_{n=0}^{\infty} c_n\phi_n(x)t^n, \qquad (c_n = \text{const.}),$$

whose explicit expressions are known, suggests in the first place Abel-summability. This was carried out by Hille [26] for Laguerre polynomials, where

$$\frac{e^{-xt}}{(1-t)^{\alpha}} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{\Gamma(n+\alpha)}{\Gamma(n+1)}\right)^{1/2} \phi_n(x) t^n, \tag{69}$$

where $|t|<1$, and where x is finite and arbitrary. Corresponding to the expansion (6), we construct the series

$$F(x, t) = \sum_{n=0}^{\infty} f_n \phi_n(x) t^n, \qquad (x \geqq 0, \text{ fixed}). \tag{70}$$

Making use of the asymptotic expression for $\phi_n(x)$ and of the expression of f_n, we show, first, that $F(x, t)$, as a function of t, is holomorphic for $|t|<1$; next, that for $t \to 1-0$, $F(x, t) \to f(x_0)$ or $[f(x_0+0)+f(x_0-0)]/2$, at every point x_0 of continuity or of discontinuity of the first kind, respectively; in other words, we show that (6) *is A-summable, if* $\int_0^{\infty} e^{-ax} x^{\alpha-1} |f(x)|\, dx$ *exists for every* $a>1/2$. By Littlewood's theorem, (6) will converge at $x=x_0$ (to the above value), if $f(x)$ is such that

$$f_n \phi_n(x_0) = O(n^{-1}). \tag{71}$$

For $x_0>0$ and $\alpha=1$, $\phi_n(x_0)=O(n^{-1/4})$; hence (71) is certainly satisfied if

$$f_n = O(n^{-3/4}), \tag{72}$$

which is shown to hold if $f(x)$ is subject to some further restrictions.

2. *Application of the Closure Equation* (15). We follow here, with considerable modification, the method developed by Stekloff [27] for the case where

$$p(x) \geqq p_0 > 0, \tag{73}$$

$p'(x)$ exists and is bounded in the finite interval (a, b),

$$f'(x) \text{ is of class } L_p^2 \text{ in } (a, b). \tag{74}$$

Write

$$f(x) = \sum_{i=0}^{n} f_i \phi_i(x) + R_n; \quad f'(x) = \sum_{i=0}^{n} f_i \phi_i'(x) + R_n', \tag{75}$$

where R_n denotes $R_n(x; f)$, and where R_n' denotes dR_n/dx. By virtue of orthogonality, we get at once

$$\int p(x)R_n\phi_i(x)dx = \begin{cases} 0, & (i \leqq n), \\ f_i, & (i > n). \end{cases} \tag{76}$$

This shows that

$$\int p(x)R_nG_n(x)dx = 0,$$

and the second relation in (75) yields the fundamental relation

$$\int p(x)R_nG_1(x)R_n'dx = \int p(x)R_nG_1(x)f'(x)dx, \tag{77}$$

($G_1(x) = g_0 + g_1x$; g_i = arbitrary constants, independent of n).

In (77) integrate by parts on the left and apply Schwartz's inequality on the right. This gives

$$\left|\int p(x)R_nG_1(x)R_n'dx\right| = \left|\frac{1}{2}\left\{p(x)R_n^2G_1(x)\Big|_{x=a}^{x=b}\right.\right.$$

$$\left.\left. - \int R_n^2\,\frac{p'(x)}{p(x)}\,p(x)G_1(x)dx - g_1\int p(x)R_n^2dx\right\}\right| \tag{78}$$

$$\leqq \int p(x)R_n^2dx \cdot \int p(x)f'^2(x)G_1^2(x)dx = o(1), \qquad (n \to \infty\,;\ \text{by (16)}).$$

Moreover, by (73) and by use of the closure equation in its form (16),

$$\int p(x)R_n^2 \cdot \frac{p'(x)}{p(x)}\,dx = o(1), \quad g_1\int pR_n^2dx = o(1), \quad (n \to \infty),$$

so that (78) leads to

$$p(x)R_n^2G_1(x)\Big|_{x=a}^{x=b} = o(1), \qquad (n \to \infty).$$

Taking here $G_1(x) = x-a$, $x-b$, we obtain

$$p(b)R_n^2(b; f) = o(1), \quad p(a)R_n^2(a; f) = o(1), \quad (n \to \infty), \tag{79}$$

which, expressed in words, states that, *under conditions* (73) *and* (74) *the expansion* (6) *converges for* $x = a, b$ *to the values* $f(a)$, $f(b)$, *respectively.**

* In some cases this alone is a sufficient basis for the conclusion that (6) converges *for all* x in (a, b).

In order to investigate R_n at any interior point, we need a further assumption as follows:

(80) $\qquad f''(x)$ is of class L_p^2.

We then add to (75) the following relation:

$$f''(x) = \sum_{i=0}^{n} f_i\phi_i''(x) + R_n'', \qquad R_n'' = \frac{d^2}{dx^2} R_n,$$

and proceeding as above, we get

$$(81) \quad \int p(x)R_nG_2(x)R_n''dx = \int p(x)R_nG_2(x)f''(x)dx = o(1), \qquad (n \to \infty).$$

Now take $G_2(x) = (x-a)(b-x)$, which is $\geqq 0$ in (a, b), and integrate by parts on the left side. Then

$$(82) \quad \int \frac{p'(x)}{p(x)} p(x)R_nR_n'G_2(x)dx + \int p(x)R_nR_n'G_2'(x)dx + \int p(x)R_n'^2G_2(x)dx = o(1), \qquad (n \to \infty).$$

Here the middle term is $o(1)$, by (78), and

$$\left| \int \frac{p'(x)}{p(x)} p(x)R_nR_n'G_2(x)dx \right|$$

$$\leqq h\gamma\left(\int \frac{p'(x)}{p(x)} R_n^2 dx \cdot \int p(x)R_n'^2 dx \right)^{1/2} = h\gamma \cdot o(1)(I_n)^{1/2},$$

$$h = \max \left| \frac{p'(x)}{p(x)} \right|, \quad \gamma = \max |G_2(x)| \text{ in } (a, b),$$

$$I_n = \int p(x)R_n'^2G_2(x)dx,$$

so that (82) becomes

$$(I_n)^{1/2} \cdot o(1) + I_n = o(1), \qquad (n \to \infty),$$

whence the important result,

$$(83) \quad I_n = \int p(x)R_n'^2(x-a)(b-x)dx = o(1), \qquad (n \to \infty).$$

Consider now the identity

$$p(x)\phi(x)R_n^2\Big|_a^x = \int_a^x [p(x)\phi(x)R_n^2]'dx = \int_a^x p'(x)\phi(x)R_n^2\,dx$$
$$+ \int_a^x p(x)\phi'(x)R_n^2\,dx + 2\int_a^x p(x)\phi(x)R_nR_n'\,dx$$
$$\equiv i_1 + i_2 + i_3, \qquad (\phi(x) = (x-a)(b-x)).$$

By the above considerations,

$$i_{1,2} = o(1) \text{ uniformly in } x, \quad (n \to \infty);$$
$$|i_3| \leqq 2\cdot\left(\frac{b-a}{2}\right)\left(\int p(x)R_n^2\,dx\cdot\int p(x)R_n'^2\phi(x)dx\right)^{1/2} = o(1),$$

whence we reach the final conclusion that

$$p(x)(x-a)(b-x)R_n^2(x) = o(1), \qquad (n\to\infty),$$

uniformly for $a \leqq x \leqq b$; that is, *under conditions* (73), (74) *and* (80), *the expansion* (6) *converges to* $f(x)$ *uniformly in the interval* $(a+\epsilon,\ b-\epsilon)$.

The method is applicable to a more general $p(x)$, of the form $(x-a)^{\alpha-1}(b-x)^{\beta-1}q(x)$, $(\alpha, \beta>0)$. We must point out, however, that the conditions imposed on $f(x)$ are far too stringent. This is compensated by the simplicity of the method, which requires neither the discussion of the coefficients f_n, nor the use of the asymptotic expression for $\phi_n(x)$. On the contrary, if we apply the above considerations to $f(x)\equiv\phi_{n+1}(x)$, we obtain directly

$$|\phi_n(c)| = (2n+1/p(c))^{1/2}(1+O(1/n)), \qquad (c=a,\ b).$$

3. *Use of the Differential Equations for the Classical OP.* These can be written as follows:

$$A(x)\phi_n''(x) + B(x)\phi_n'(x) + C_n\phi_n(x) = 0, \tag{84}$$

where A and B are polynomials of degree $\leqq 2$, independent of n, and where C_n denote constants.

$$\begin{aligned}
&\text{(J)}\quad (1-x^2)\phi_n''(x) + [\alpha-\beta-(\alpha+\beta)x]\phi_n'(x)\\
&\qquad\qquad + n(n+\alpha+\beta-1)\phi_n(x) = 0,\\
&\text{(L)}\quad x\phi_n''(x) + (\alpha-x)\phi_n'(x) + n\phi_n(x) = 0,\\
&\text{(H)}\quad \phi_n''(x) - 2x\phi_n'(x) + 2n\phi_n(x) = 0, \qquad (n = 0, 1, \cdots).
\end{aligned} \tag{85}$$

From this we learn at once, by differentiation, that $\phi_n'(x)$ *is an OP of the same class, with* $p(x)$ *replaced by* $p_1(x)=A(x)p(x)$. Denoting the normalized $\phi_n'(x)$ by $\psi_{n-1}(x)$, we derive, by a simple computation, the following important relation which holds true for all classical *OP*:

$$\phi_n'(x) = (C_n)^{1/2}\psi_{n-1}(x), \qquad (n = 1, 2, \cdots), \tag{86}$$

where

$$\int p_1(x)\psi_m(x)\psi_n(x)dx = \delta_{mn}, \tag{87}$$

where $m, n=0, 1, \cdots$, and where $p_1(x)=A(x)p(x)$. Furthermore, it is readily seen that for all classical *OP*

$$(A(x)p(x))' = B(x)p(x), \qquad A(x)p(x) = 0 \text{ for } x = a, b. \tag{88}$$

Assuming now the existence in (a, b) of $f'(x)$ of the class L_p^2, (88) enables us to treat the coefficient f_n as follows. We write

$$\begin{aligned} C_n f_n &= \int p(x)f(x)C_n\phi_n(x)dx \\ &= \int p(x)f(x)[-A\phi_n''(x) - B\phi_n'(x)]dx, \end{aligned}$$

whence, by integration by parts, in virtue of (86) and (88),

$$f_n = \frac{f_{n-1}'}{(C_n)^{1/2}}, \tag{89}$$

where

$$f_{n-1}' = \int p_1(x)f'(x)\psi_{n-1}(x)dx, \text{ that is, } f'(x) \sim \sum_{n=0}^{\infty} f_n'\psi_n(x).$$

We have now

$$|f_n\phi_n(x)| = \left| f_{n-1}' \cdot \frac{\phi_n(x)}{(C_n)^{1/2}} \right| \leqq \frac{1}{2}\left\{ f_{n-1}'^2 + \frac{\phi_n^2(x)}{C_n} \right\}.$$

Here the series $\sum_{n=1}^{\infty} f_{n-1}'^2$ certainly converges (closure), and

$$
(90)\quad
\begin{aligned}
&\text{(J)} \quad \phi_n^2(x)/C_n = O(n^{-2}), && (-1+\epsilon \leqq x \leqq 1-\epsilon);\\
&\text{(L)} \quad \phi_n^2(x)/C_n = O(n^{-3/2}), && \\
& && (0 < \epsilon \leqq x \leqq A;\ A \text{ finite, fixed arbitrarily});\\
&\text{(H)} \quad \phi_n^2(x)/C_n = O(n^{-3/2}), && (|x| \leqq A).
\end{aligned}
$$

Hence, *the expansion* (6) *in series of classical OP converges uniformly and absolutely in the respective intervals as given in* (90), *if* $f'(x)$ *is of the class* L_p^2 *in* (a, b).

The same results could be obtained, without making use of the asymptotic expression for $\phi_n(x)$ (necessary to establish (90)), if we combine the use of the differential equation with that of closure as in the previous section (this would eliminate the introduction of $f''(x)$).

4. *Comparison Method.* The problem before us is: Given $p_1(x) \leqq p_2(x)$ in (a, b), where $p_1(x)$ and $p_2(x)$ are two weight-functions, find a corresponding qualitative relation between the partial sums of the two corresponding expansions (6) of the same function $f(x)$. This problem was solved by Szegö [28]. The basis of his analysis is the solution of the following *two-parameter* extremal problem.

Given in $(-1, 1)$ *a weight-function* $p(x)$ *and a function* $F(x)$ *such that* $F(x)/p(x)$ *and* $F^2(x)/p(x)$ *are both of class* L, *find* $\max \{\lambda G_n(\xi) + \mu \int_{-1}^{1} F(x) G_n(x)\,dx\}^2$, ($\lambda, \mu$ *parameters*, ξ *a fixed point inside* $(-1, 1)$), *for all* $G_n(x)$ *satisfying the condition* $\int_{-1}^{1} p(x) G_n^2(x)\,dx = 1$.

Introduce the *OP* corresponding in $(-1, 1)$ to $p(x)$ and write

$$
(91)\quad
\begin{aligned}
G_n(x) &= \sum_{i=0}^{n} \gamma_i \phi_i(x), \ \gamma_i \text{ being constants, with } \sum_{i=0}^{n} \gamma_i^2 = 1.\\
\frac{F(x)}{p} &\sim \sum_{i=0}^{\infty} \alpha_i \phi_i(x), \qquad \alpha_i = \int_{-1}^{1} F(x)\phi_i(x)\,dx,\\
\frac{F(x)}{p(x)} &= \sum_{i=0}^{n} \alpha_i \phi_i(x) + R_n \equiv S_n\left(x;\ \frac{F}{p}\right) + R_n,\\
U_n(\lambda, \mu) &\equiv \left\{\lambda G_n(\xi) + \mu \int_{-1}^{1} F(x) G_n(x)\,dx\right\}^2\\
&= \left\{\sum_{i=0}^{n} \gamma_i [\lambda \phi_i(\xi) + \mu \alpha_i]\right\}^2, \qquad \sum_{i=0}^{n} \gamma_i^2 = 1.
\end{aligned}
$$

Then, by Cauchy's inequality and (13),

$$M_n(\lambda, \mu; p) \equiv \max U_n(\lambda, \mu) = \sum_{i=0}^{n} (\lambda\phi_i(\xi) + \mu\alpha_i)^2,$$

$$M_n(\lambda, \mu; p) = \lambda^2 K_n(\xi) + 2\lambda\mu \sum_{i=0}^{n} \alpha_i\phi_i(\xi) + \mu^2 \sum_{i=0}^{n} \alpha_i^2$$

$$= \lambda^2 K_n(\xi) + 2\lambda\mu S_n\left(\xi; \frac{F}{p}\right) + \mu^2 U_n\left(\frac{F}{p}\right), \tag{92}$$

$$U_n\left(\frac{F}{p}\right) \equiv \sum_{i=0}^{n} \alpha_i^2 \leqq \int \frac{F^2(x)}{p(x)}\, dx \equiv U(p).$$

On the basis of the condition $\int_{-1}^{1} p(x) G_n^2(x)dx = 1$, we readily verify that

$$\begin{aligned} &p_1(x) \leqq p(x) \leqq p_2(x) \text{ in } (-1, 1) \text{ implies} \\ &M_n(\lambda, \mu; p_2) \leqq M_n(\lambda, \mu; p) \leqq M_n(\lambda, \mu; p_1). \end{aligned} \tag{93}$$

Hence, the difference $M_n(\lambda, \mu; p) - M_n(\lambda, \mu; p_2)$, which is a quadratic form in λ, μ, is positive for all real λ, μ. Expressing the fact that its discriminant is non-positive, we get, replacing $F(x)/p(x)$ by $f(x)$, of class L_p^2, and writing $S_n(x; p), K_n(x; p), U_n(p), \cdots$,

$$\begin{aligned} &p(x) \leqq p_2(x) \text{ in } (-1, 1) \text{ implies} \\ &[S_n(\xi; p) - S_n(\xi; p_2)]^2 \leqq U_n(p) \cdot [K_n(\xi; p) - K_n(\xi; p_2)] \\ &\qquad \leqq U(p)[K_n(\xi; p) - K_n(\xi; p_2)], \end{aligned} \tag{94}$$

where

$$U(p) = \int p(x) f^2(x) dx.$$

This is the desired relation between $S_n(x; p)$ and $S_n(x; p_2)$. Incidentally we have found here before (94) two other important relations of the same kind:

$$p_1(x) \leqq p(x) \leqq p_2(x) \text{ in } (-1, 1) \tag{95}$$

implies

$$K_n(\xi; p_1) \leqq K_n(\xi; p) \leqq K_n(\xi; p_2), \quad U_n(p_1) \leqq U_n(p) \leqq U_n(p_2).$$

It now follows that

$$p_1(x) \leqq p(x) \leqq p_2(x) \text{ in } (-1, 1) \tag{96}$$

implies

$$(S_n(\xi; p_1) - S_n(\xi; p_2))^2 \leqq U(p)[K_n(\xi; p_1) - K_n(\xi; p_2)].$$

Thus, if we succeed in choosing $p_1(x)$ and $p_2(x)$ in such a manner that

$$\begin{aligned} &p_1(x) \leqq p(x) \leqq p_2(x) \text{ in } (-1, 1), \\ &K_n(\xi; p_1) - K_n(\xi; p_2) = o(1), \ (n \to \infty\,; \xi \text{ given inside } (-1, 1)), \end{aligned} \tag{97}$$

then (94) *leads directly to an equiconvergence theorem*, namely,

$$\lim_{n\to\infty} [S_n(\xi; p) - S_n(\xi; p_2)] = 0. \tag{98}$$

Szegö shows that such a choice of $p_1(x)$ and $p_2(x)$ is possible, if

$$\begin{cases} p(x) = (1 + x)^{\alpha-1}(1 - x)^{\beta-1}q(x), \text{ with } 1/2 \leqq \alpha, \beta \leqq 3/2, \\ q(x) \text{ bounded and positive, } p(x) \text{ is } R\text{-integrable in } (-1, 1), \end{cases} \tag{99}$$

provided *at the interior point ξ under consideration $p(x)$ is continuous with its first and second derivatives.* We then can take

$$p_1(x) = \frac{(1 - x^2)^{1/2}}{P_1(x)}, \quad p_2(x) = \frac{1}{(1 - x^2)^{1/2}P_2(x)}, \tag{100}$$

where $P_1(x)$ and $P_2(x)$ are polynomials greater than zero in $(-1, 1)$; and we can choose the polynomials $P_1(x)$ and $P_2(x)$ so that, in addition to the first condition (97),

$$0 \leqq \int_{-1}^{1} \frac{\log p_2(x) - \log p(x)}{(x - \xi)^2} \cdot \frac{dx}{(1 - x^2)^{1/2}} < \epsilon; \tag{101}$$

$$(p_1(\xi) = p(\xi) = p_2(\xi)).$$

Szegö then obtains an explicit expression for $\phi_n(x; p_{1,2})$ (from a certain n on) using Fejér's theorem on the trigonometric representation of positive polynomials [29], and by means of Darboux's formulas shows that the second condition (97) is also satisfied. Incidentally we find in this way an asymptotic expression for $K_n(x; p)$ and $\phi_n(x; p)$. Thus, (98) is established.

But this is an intermediate step only. The desired equiconvergence theorem in its final form is

$$S_n(\xi; p) - \frac{\sigma_n(\xi)}{p(\xi)(1-\xi^2)^{1/2}} \equiv o(1), \qquad (n \to \infty), \tag{102}$$

where $\sigma_n(\xi)$ represent the nth partial sum at the interior point $\xi = \cos\theta_0$ of the Fourier cosine-series expansion of the function $p(\cos\theta) f(\cos\theta) |\sin\theta|$. This is readily established, by first showing that (102) holds true for $p_2(x)$ (where the explicit expression of $\phi_n(x; p_2)$ is known) and then again using (96). Taking $q(x) \equiv 1$, we obtain the equiconvergence theorem for Jacobi polynomials, with $1/2 \leqq \alpha, \beta \leqq 3/2$.

IV. Illustration of the General Methods Discussed Above by Means of Legendre Polynomials.

The Legendre polynomials

$$\phi_n(x; 1) = \left(\frac{2n+1}{2}\right)^{1/2} P_n(x),$$

where

$$\frac{1}{(1-2tx+t^2)^{1/2}} = \sum_{n=0}^{\infty} P_n(x) t^n,$$

have, among others, the following properties:

$$\phi_n(-x) \equiv (-1)^n \phi_n(x); \quad |\phi_n(x)| \leqq |\phi_n(+1)| = \left(\frac{2n+1}{2}\right)^{1/2},$$

$(-1 \leqq x \leqq 1)$. Applying the preceding general methods to the expansion (6) of a given function $f(x)$ according to Legendre polynomials, we obtain the following results regarding its convergence.

(i) *Case of continuous functions.* From the estimate of $f_n\phi_n(x)$ we learn that (6) converges absolutely and uniformly in $(-1, 1)$ if $f'(x)$ satisfies therein a Lipschitz condition of order $> 1/2$. From the estimate of $R_n(x; f)$ we learn that (6) converges uniformly in any fixed interval $(-1+\epsilon, 1-\epsilon)$, if $f(x)$ satisfies in $(-1, 1)$ a Dini-Lipschitz condition, and uniformly in the whole interval $(-1, 1)$, if $f'(x)$ is therein continuous.

(ii) *Use of integral equations.* The kernel here is [10]

$$K(x, y) = \begin{cases} \log(1-y) + \log(1+x) + 1 - 2\log 2, \\ \qquad\qquad (-1 \leqq y \leqq x \leqq 1), \\ \log(1+x) + \log(1-y) + 1 - 2\log 2, \\ \qquad\qquad (-1 \leqq x \leqq y \leqq 1). \end{cases}$$

The series (6) converges absolutely and uniformly in $(-1, 1)$ if $f(x), f'(x)$, and $f''(x)$ are therein continuous.

(iii) *Application of the Rademacher-Menchoff theorem.* The series (6) converges almost everywhere in $(-1, 1)$ if $f(x)$ is of bounded variation in $(-1, 1)$, or if

$$|f(x') - f(x'')| \cdot |\log|x' - x''||^{1+\epsilon} < \text{const.},$$
$$(-1 \leqq x', x'' \leqq 1).$$

(iv) *Use of Lebesgue constants.* Here

$$\rho_n(x) \sim \log n, \ (-1 < x < 1), \quad \rho_n(\pm 1) \sim n^{1/2}.$$

The series (6) converges uniformly in $(-1+\epsilon, 1-\epsilon)$ or in $(-1, 1)$, if $f(x)$ satisfies in $(-1, 1)$ a Dini-Lipschitz condition or a Lipschitz condition of order $>1/2$, respectively.

(v) *Use of the asymptotic expression of* $\phi_n(x)$. At any interior point $x = \cos\theta$, (6) behaves, regarding convergence or divergence, (a) like the ordinary Fourier series expansion of $f(\cos\theta)$, if $f(x)$ is of the class L^2 in $(-1, 1)$; (b) like the ordinary Fourier series expansion of $f(\cos\theta)|\sin\theta|$, if $f(x)$ is of the class L in $(-1, 1)$ and $\int_{-1}^{1}(1-x^2)^{-1/4}|f(x)|\,dx$ exists.

We may add that if we wish to discuss the effect of singularities of $f(x)$ at the end points $x = \pm 1$ on the convergence (or summability) properties of (6), the zero-series (p. 65) proves the most effective tool.

We see that the weakest method is that of integral equations, the strongest one, that of asymptotic expressions. Note that the latter enter explicitly or implicitly in many other methods dealing with the convergence of (6). The above considerations apply not only to ordinary convergence, but also to summability of (6), as was pointed out above. They also remain valid in many parts if we replace $p(x)dx$ by the more general $d\psi(x)$, where $\psi(x)$ is monotonic non-decreasing in (a, b), and use Stieltjes integrals.

We have had to omit in our discussion many other interesting topics. The most important are those that follow.

(i) *Gibbs' phenomenon*, familiar from the theory of ordinary Fourier series.

(ii) *The problem of Cantor.* Can we have $\sum_{n=0}^{\infty} a_n \phi_n(x) = 0$, $(a \leqq x \leqq b)$, without having $a_n = 0$ for all n? This is evidently equivalent to the problem of *uniqueness* of the expansion of a given $f(x)$ in series of orthogonal polynomials: can we have $\sum_{n=0}^{\infty} a_n \phi_n(x) = f(x)$, $\sum_{n=0}^{\infty} b_n \phi_n(x) = f(x)$ for certain x, without having $a_n = b_n$ for all n?

(iii) *The problem of Dubois-Reymond.* Assume the existence of $\int p(x) f(x) \phi_n(x) dx \equiv f_n$, $(n \geqq 0)$. If $\sum_{n=0}^{\infty} a_n \phi_n(x) = f(x)$ on a specified set $E \subset (a, b)$, can we conclude that $a_n = f_n$ for all n? This is equivalent to the problem *of term by term integration of the above series.*

These problems and many others on the general *OP* offer a vast and fruitful field of research. In some special cases (polynomials of Legendre, symmetric Jacobi polynomials), interesting results have been obtained by Plancherel and Kogbetliantz.

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17. D. Menchoff, Fundamenta Mathematicae, vol. 4 (1923), pp. 82–105.

18. I. Natanson, *On the convergence of series according to orthogonal polynomials*, Comptes Rendus, Académie des Sciences URSS, (1934), pp. 209–212 (in Russian.)

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21. Jacques Chokhate (J. Shohat), *Sur la sommation de certaines séries des fonctions intégrables. Application aux fonctions orthogonales*, Comptes Rendus, vol. 188 (1929), pp. 1475–1477.

22. E. W. Hobson, *The Theory of Spherical and Elliptical Harmonics*, 1931.

23. A. Haar, Mathematische Annalen, vol. 78 (1918), pp. 121–136.

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26. Einar Hille, *On Laguerre polynomials*, First Note, Second Note, Third Note, Proceedings National Academy of Sciences, vol. 12 (1926), pp. 261–269, pp. 348–352.

27. W. Stekloff, *Sur une application de la théorie de fermeture au problème du développement des fonctions arbitraries en séries procédant suivant les polynomes de Tchebycheff*, Mémoires, Académie Impériale des Sciences de Petrograd, Classe Physico-Mathématique, vol. 33 (1914), pp. 1–58.

28. G. Szegö, *Ueber die Entwickelung einer willkürlicher Funktion nach den Polynomen eines Orthogonalsystems*, Mathematische Zeitschrift, vol. 12 (1922), pp. 61–94.

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THE UNIVERSITY OF PENNSYLVANIA

THE CATEGORY OF THE CLASS LIP (α, p)

BY E. S. QUADE

A function $x(s)$ is said to belong to the class $\mathrm{Lip}(\alpha, p)$ on the interval (a, b) provided

$$\| x(s+h) - x(s) \| \equiv \left(\int_a^b | x(s+h) - x(s) |^p ds \right)^{1/p} = O(h^\alpha),$$

where $0 < \alpha \leqq 1$.

There exist continuous functions which belong to no class $\mathrm{Lip}(\alpha, p)$. Indeed if $x(s) \subset \mathrm{Lip}(\alpha, p)$, then the Fourier coefficients of $x(s)$, a_n, b_n, are $O(n^{-\alpha})$. Now a continuous function may be constructed* such that $|a_{n_i}| > 1/\log n_i$ for an infinite set of values $\{n_i\}$. Then for such a function

$$\frac{|a_{n_i}|}{n_i^{-\alpha}} > \frac{n_i^\alpha}{\log n_i} \neq O(1),$$

that is, $a_n \neq O(n^{-\alpha})$ and hence the continuous function with the Fourier coefficients a_n belongs to no class $\mathrm{Lip}(\alpha, p)$.

We prove the following theorem.

THEOREM. *The subset E of L_p, $p \geqq 1$, which is $\sum \mathrm{Lip}(\alpha, p)$ for $0 < \alpha \leqq 1$, is of the first category in L_p.*

We employ a method of proof used by S. Banach.† We take the interval $(0, 1)$ as the fundamental interval and assume the functions to be periodic with the period one. Let E_{nm} be the set of all $x(s) \subset L_p$ such that

$$\int_0^1 | x(s+h) - x(s) |^p ds \leqq n^p | h |^{p/m}, \qquad (n, m = 1, 2, \cdots).$$

The sets E_{nm} are closed. For, let $x_i(s) \to x_0(s)$ in L_p. Set

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* W. Randels, *A remark on Fourier series of continuous functions*, American Mathematical Monthly, vol. 40 (1933), pp. 97–99. See also an article by O. Szász, to appear soon in the same journal.

† *Über die Baire'sche Kategorie gewisser Funktionenmengen*, Studia Mathematica, vol. 3 (1931), pp. 174–179.

$$y_i(s) = x_i(s + h) - x_i(s),$$
$$y_0(s) = x_0(s + h) - x_0(s),$$

where h is fixed but arbitrary. Then

$$\|y_i - y_0\| \leqq \|x_i(s + h) - x_0(s + h)\| + \|x_i(s) - x_0(s)\|$$
$$= 2\|x_i(s) - x_0(s)\| \to 0.$$

But $\|y_i - y_0\| \to 0$ implies that $\|y_i\| \to \|y_0\|$, that is,

$$\int_0^1 |x_0(s + h) - x_0(s)|^p ds \leqq n^p |h|^{p/m}.$$

Moreover $E \subset \sum_{n,m=1}^{\infty} E_{nm}$. For, if $x_0(s) \subset E$, then for some value α_0, $x_0(s) \subset \text{Lip}(\alpha_0, p)$; that is, there exists a number M such that

$$\int_0^1 |x_0(s + h) - x_0(s)|^p ds \leqq M |h|^{\alpha_0 p}.$$

To complete the proof we have only to show that every set E_{nm} is non-dense. Suppose, if possible, that E_{NM} were not non-dense. Then, since E_{NM} is closed, it contains a sphere K. Let $\omega(s) \subset K \subset E_{NM}$ be the center of the sphere and $r > 0$ the radius. Let $g(s) \subset L_p$ be a function of E. Since when $g(s) \subset E$, $c \cdot g(s) \subset E$, where c is a constant not zero, we may assume $\|g\| < r$. Also

$$\|g(t + h) - g(t)\| > 2N |h|^{1/M}.$$

Set $z(s) = \omega(s) + g(s)$. Then $z(s) \subset L_p$ and

$$\|z(t + h) - z(t)\| \geqq \|g(t + h) - g(t)\| - \|\omega(t + h) - \omega(t)\|$$
$$> 2N |h|^{1/M} - N |h|^{1/M} \geqq N |h|^{1/M};$$

that is, $z(s)$ not $\subset E_{NM}$. But $\|z - \omega\| = \|g\| < r$; this means $z(s) \subset K \subset E_{NM}$, a contradiction.

In exactly the same manner we may prove the following result.

The subset EC of the space C of continuous functions is of the first category in C.

BROWN UNIVERSITY

TERNARY ARITHMETICAL IDENTITIES*

BY E. T. BELL

1. *Introduction.* Two sets of identities are said to be equivalent when each set implies the other. By the method of paraphrase,† an elliptic theta identity is equivalent to one or more identities in functions with integer variables, the functions usually being subject to restrictions of parity (evenness or oddness in sets of variables). On account of their importance for class number relations‡ and other questions concerning quadratic forms in two or more variables, it is of interest to have the complete set of arithmetical identities equivalent to all the identities implied by the addition theorems for the thetas and the transformations of the first and second orders that are bilinear in thetas and doubly periodic functions of the second kind. Changes of q into $-q$, or increase of the variables by integer multiples of half periods in one of the latter identities, produce arithmetical equivalents obtainable immediately from the arithmetical equivalent of the original identity by proper specialization of the parity functions. Hence it is unnecessary to discuss such derived identities. Omitting these, we find precisely four identities to be paraphrased:

$$(1) \qquad \sum \vartheta_1(-u+v+w)\phi_{100}(2w, -2v, q^2) = 0;$$

$$(2) \qquad \sum \vartheta_0(-u+v+w)\phi_{111}(2w, -2v, q^2) = 0;$$

$$(3) \qquad \sum \vartheta_1(-u+v+w)\phi_{122}(w, -v, q^{1/2}) = 0;$$

$$(4) \qquad \sum \vartheta_3(-u+v+w)\phi_{111}(w, -v, q^{1/2}) = 0;$$

where the sum refers to the three products obtained in each case from the one written by the substitutions 1, uvw, uwv; and

$$\phi_{rst}(x, y) \equiv \phi_{rst}(x, y, q) \equiv \vartheta_1'\vartheta_r(x+y)/[\vartheta_s(x)\vartheta_t(y)],$$

the parameter in the thetas being q. The arithmetical equiva-

* Presented to the Society, October 27, 1934.

† E. T. Bell, Transactions of this Society, vol. 22 (1921), pp. 1–30, 198–219. Cited as B.

‡ J. V. Uspensky, Bulletin de l'Académie des Sciences de Russie, 1925, pp. 599–620, 763–784; 1926, pp. 25–38, 175–196, 327–348. Cited as U.

lents of (1)–(4) are stated in §2, and several equivalents of these are indicated in §3.

To prove (1)–(4), we take

$$A = \vartheta_a(x), \qquad B = \vartheta_a(y), \qquad C = \vartheta_a(z),$$
$$X = \vartheta_b(x), \qquad Y = \vartheta_b(y), \qquad Z = \vartheta_b(z), \qquad a \neq b,$$

in the identity

$$\begin{vmatrix} A & B & C \\ X & Y & Z \\ X & Y & Z \end{vmatrix} = 0;$$

reduce the results, for $(a, b) = (1, 2), (0, 3), (1, 0), (3, 2)$, by the addition theorems for the thetas; introduce the doubly periodic functions of the second kind by dividing throughout by the appropriate product of the thetas; and finally make the change of variables

$$x = -u+v+w, \qquad y = u-v+w, \qquad z = w+v-w.$$

The addition theorems used are

$$\vartheta_1(x)\vartheta_2(y) - \vartheta_2(x)\vartheta_1(y) = 2\vartheta_0(x+y, q^2)\vartheta_1(x-y, q^2),$$
$$\vartheta_0(x)\vartheta_3(y) - \vartheta_3(x)\vartheta_0(y) = 2\vartheta_1(x+y, q^2)\vartheta_1(x-y, q^2),$$
$$\vartheta_1(x)\vartheta_0(y) - \vartheta_0(x)\vartheta_1(y) = \vartheta_2\left(\frac{x+y}{2}, q^{1/2}\right)\vartheta_1\left(\frac{x-y}{2}, q^{1/2}\right),$$
$$\vartheta_2(x)\vartheta_2(y) - \vartheta_2(x)\vartheta_3(y) = \vartheta_1\left(\frac{x+y}{2}, q^{1/2}\right)\vartheta_1\left(\frac{x-y}{2}, q^{1/2}\right).$$

The arithmetical equivalents of (1)–(4) are summarized at the end of §3.

2. *Arithmetical Functions.* An arithmetical function is one which is finite and single-valued for all sets of integer values of the variables. Let $f(x, y, z), \cdots, L(x, y, z)$ be arithmetical functions of x, y, z subject only to the following restrictions, in which x, y, z denote integers.

(5) $f(x, y, z) = f(-x, -y, -z)$.

(6) $g(x, y, z) = -g(-x, -y, -z), \qquad g(0, 0, 0) = 0.$

(7) $F(x, y, z) = F(y, z, x) = F(z, x, y) = F(-x, -y, -z)$.

(8) $G(x, y, z) = G(y, z, x) = G(z, x, y) = -G(-x, -y, -z)$, $G(0, 0, 0) = 0$.

(9) $H(x, y, z) = H(y, z, x) = H(z, x, y)$.

(10) $K(x, y, z) = K(y, z, x) = K(z, x, y) = K(x, z, y)$.

(11) $L(x, y, z) = L(y, z, x) = L(z, x, y) = -L(x, z, y)$.

Thus $F(x, y, z)$ is any even arithmetical function of x, y, z which belongs to the cyclic group on x, y, z; $G(x, y, z)$ is any odd arithmetical function of x, y, z belonging to the same group; $H(x, y, z)$ is any arithmetical function of x, y, z belonging to this group; $K(x, y, z)$ is any arithmetical function of x, y, z belonging to the symmetric group on x, y, z; and $L(x, y, z)$ is any alternating arithmetical function of x, y, z. Note that these functions need not be defined when x, y, z are not all integers.

We can now state arithmetical equivalents of (1)–(4). The letters x, y, z denote variable integers, m, n constant integers; $(-1|x) \equiv (-1)^{(x-1)/2}$; $\epsilon(a) = 1$, or 0, according as a is or is not, the square of an integer >0. The equivalents are numbered correspondingly to (1)–(4); thus (I_1) and (1) are equivalent, etc.

(I_1) $m = x^2 + 4yz$; $x \gtrless 0$, $y > 0$, $z > 0$; x, y, z odd:

$$\sum(-1|x)F(-x, x-2z, x+2y) = 0.$$

(II_1) or (IV_1) $n = x^2 + yz$; $x \gtrless 0$, $y > 0$, $z > 0$:

$$G(-x, x-z, x+y) = \epsilon(n)\left[G(-n^{1/2}, n^{1/2}, 0) + \sum_{r=1}^{n^{1/2}-1}\left\{G(-n^{1/2}, n^{1/2}, r) - G(-n^{1/2}, r, n^{1/2})\right\}\right].$$

(II_2) $m = x^2 + 4yz$; $x \gtrless 0$, odd; $y > 0$, $z > 0$:

$$G(-x, x-2z, x+2y) = \epsilon(m)\sum_{r=1}^{(m^{1/2}-1)/2}\left[G(-m^{1/2}, m^{1/2}, 2r-1) - G(-m^{1/2}, 2r-1, m^{1/2})\right].$$

(III_1) $m = x^2 + 4yz$; $x \gtrless 0$, odd; $y > 0$, $z > 0$:

$$-\sum(-1|x)(-1)^{y+z}F(-x, x-2z, x+2y) = \epsilon(m)\sum_{r=1}^{(m^{1/2}-1)/2}(-1)^r\left[F(-m^{1/2}, m^{1/2}, 2r-1) - F(-m^{1/2}, 2r-1, m^{1/2})\right].$$

Thus (II_1) is equivalent to (4); (II_1) and (II_2) are together equivalent to (2); and it is evident that (I_1) is the special case $m \equiv 5 \bmod 8$ of (III_1). If $m \equiv 5 \bmod 8$ in (II_2), we get

$$(II_3) \quad m = x^2+4yz;\ x \gtrless 0, \text{ odd};\ y > 0,\ z > 0;\ m \equiv 5 \bmod 8:$$
$$\sum G(-x,\ x-2z,\ x+2y) = 0.$$

It will be seen in §4 that (II_1) is equivalent to the fundamental identity of Uspensky (U, loc. cit.), which was shown in a previous paper* to be equivalent to (2) written with different arguments (an unsymmetrical form.)

To indicate the proofs we transform (I_1), (III_1) into their equivalents in terms of f defined in (5), and (II_1), (II_2) into their equivalents in terms of g defined in (6).

The summations in what follows extend over some set, which need not be specified, of triples of integers x, y, z. The constants c_{xyz} do not involve the functions $f, \cdots, L$ occurring in a particular sum.

We see first that

$$(12) \qquad \sum c_{xyz}[f(x, y, z) + f(y, z, x) + f(z, x, y)] = 0,$$

$$(13) \qquad \sum c_{xyz}F(x, y, z) = 0$$

are equivalent. For (5) is the only restriction upon f and, by (7), $F(x, y, z) = F(-x, -y, -z)$. Hence f may be replaced by F in (12). Reduction of the result by (7) gives (13). Thus (12) implies (13). Conversely, $f(x, y, z)+f(y, z, x)+f(z, x, y)$, considered as a function, say $F'(x, y, z)$, of x, y, z, satisfies all the conditions (7) on $F(x, y, z)$. Hence (13) implies (12). Thus (12), (13) are equivalent. In the same way

$$(14) \qquad \sum c_{xyz}[g(x, y, z) + g(y, z, x) + g(z, x, y)] = 0,$$

$$(15) \qquad \sum c_{xyz}G(x, y, z) = 0$$

are seen to be equivalent.

Restating (I_1), (III_1) by means of (12), (13) as their f-equivalents we get them in the forms into which (1), (3) paraphrase immediately by the method of the paper (B). As the details are all simple routine which has been exemplified many times in that paper and in (B_2), we shall omit them. Similarly for (II_1), (II_2) and (14), (15).

* E. T. Bell, this Bulletin, vol. 32 (1933), pp. 682–687. Cited as B_2.

3. *Arithmetical Equivalents.* By (7), (9) we may replace $F(x, y, z)$ by $H(x, y, z)+H(-x, -y, -z)$ in (13); and by (8), (9), $G(x, y, z)$ may be replaced by $H(x, y, z)-H(-x, -y, -z)$ in (15). Hence (13), (15) imply

$$\sum c_{xyz}[H(x, y, z) + H(-x, -y, -z)] = 0, \tag{16}$$

$$\sum c_{xyz}[H(x, y, z) - H(-x, -y, -z)] = 0, \tag{17}$$

respectively. Considered as a function, say $H'(x, y, z)$, of x, y, z, $F(x, y, z)+G(x, y, z)$ satisfies the conditions (9) on $H(x, y, z)$. Hence we may replace $H(x, y, z)$ by $F(x, y, z)+G(x, y, z)$ in (16), (17). By (7), (8) the results reduce to (13), (15). Thus (13), (16) are equivalent, and likewise for (15), (17).

We next see that

$$\sum c_{xyz}[H(x, y, z) - H(x, z, y)] = 0, \tag{18}$$

$$\sum c_{xyz}L(x, y, z) = 0 \tag{19}$$

are equivalent. The implication of (18) by (19) follows from (11), (9). To see that (18) implies (19), we note that $L(x, y, z)+K(x, y, z)$ satisfies the conditions (9). Hence we may replace $H(x, y, z)$ by $L(x, y, z)+K(x, y, z)$ in (18). Reducing by (10), (11), we get (19).

Application of the foregoing equivalences to the identities in §2 gives their equivalents. Note that in the identities in §2 the sign of x may be changed in any of the functions, and y, z may be interchanged. In this way we find that

$$\sum(-1 \mid x)\ L(-x, x-2z, y+2z) = 0 \tag{I_2}$$

is equivalent to (I_1), for the partition as in (I_1), and that (II_3) is equivalent to

$$\sum L(-x, x-2z, y+2z) = 0 \tag{II_4}$$

for the same partition. Hence, separating x modulo 4, and combining (I_2), (II_4), we see that (I_1) and (II_3) together are equivalent to the pair

(20) $m = x^2 + 4yz;\ x \gtrless 0,\ y > 0, z > 0;\ x, y, z$ odd; $x \equiv j \bmod 4$:

$$\sum L(-x, x-2z, x+2y) = 0,$$

where j is either 1 or -1.

For (II_1) we have the equivalents (21) (II_5) below, in which the partition is as in (II_1).

$$\sum[H(-x, x-z, x+y) - H(-x, x+z, x-y)]$$

$$= \epsilon(n)\Big[H(-n^{1/2}, n^{1/2}, 0) - H(-n^{1/2}, 0, n^{1/2})$$

$$(21) \qquad + \sum_{r=1}^{n^{1/2}-1}\Big\{H(-n^{1/2}, n^{1/2}, r) - H(-n^{1/2}, r, n^{1/2})$$

$$-H(n^{1/2}, -n^{1/2}, -r) + H(n^{1/2}, -r, -n^{1/2})\Big\}\Big];$$

$$L(-x, x-z, x+y) = \epsilon(n)\Big[L(-n^{1/2}, n^{1/2}, 0)$$

$$(II_5) \qquad + \sum_{r=1}^{n^{1/2}-1}\Big\{L(-n^{1/2}, n^{1/2}, r) - L(n^{1/2}, -n^{1/2}, -r)\Big\}\Big].$$

Omitting the H-equivalents of (II_2), (III_1), we pass to the L-equivalent; the partition is as in (II_2), (III_1):

$$-\sum\{(-1)^{y+z}(-1\,|\,x) \mp 1\}L(-x, x-2z, x+2y)$$

$$(II_6) \qquad = \epsilon(m)\sum_{r=1}^{(m^{1/2}-1)/2}[\{(-1)^r \pm 1\}L(-m^{1/2}, m^{1/2}, 2r-1)$$

$$+ \{(-1)^r \mp 1\}L(m^{1/2}, -m^{1/2}, -2r+1)],$$

the upper or the lower signs being taken throughout. This is equivalent to (II_2), (III_1).

The L-equivalents appear to be the most convenient. In summary for these, (1)–(4) are equivalent to (20), (II_5), (II_6) of this section, L being defined by (11). As will be indicated next, (II_5) is equivalent to Uspensky's identity.

4. *Conclusion.* Let $k(x, y, z)$ be an arithmetical function of x, y, z satisfying the conditions

$$(22) \quad k(x, y, z) = -k(-x, y, z) = k(x, -y, -z), \quad k(0, y, z) = 0.$$

Then k satisfies the conditions (6), and hence g may be replaced by k in the g-equivalent of (II_5) or (II_1), since the last pair are equivalent. Again, if x, y, z in (22) be replaced by any linear homogeneous functions of x, y, z with constant integer coefficients not all zero, the transformed $k(x, y, z)$ is an arithmetical

function of x, y, z satisfying (22). Conversely, if the determinant of the transformation is ± 1, the new k-identity implies the old (the restriction that the determinant be ± 1, and not merely $\neq 0$, which would suffice for the existence of the inverse transformation, is necessary to ensure that the transformed functions shall be arithmetical as defined here). These conditions are satisfied by the transformation

$$\begin{matrix} x \to a_1x + a_2y + a_3z, \\ y \to b_1x + b_2y + b_3z, \\ z \to c_1x + c_2y + c_3z, \end{matrix} \quad \left\| \begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix} \right\| = \left\| \begin{matrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 2 & 1 & 1 \end{matrix} \right\|,$$

which reduces the g-equivalent of (II_1) immediately to Uspensky's form. Uspensky's proof was entirely elementary. Mordell* gave an elementary proof of an equivalent of Uspensky's identity, and Oppenheim† gave an elementary proof of an identity which I showed‡ to be equivalent to Uspensky's. It may be mentioned that the 48 arithmetical expansions of functions $\phi_{rst}(x, y)$ which are not doubly periodic of the second kind, and which will be published shortly in the American Journal, lead at once to similar identities concerning quadratic forms in seven variables, for example $x^2+tv+yz+wu$. The general case concerns forms in $4s+3$ variables; the identities of the present paper correspond to $s=0$.

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* L. J. Mordell, Journal of the London Mathematical Society, vol. 4 (1929), pp. 291–296.

† A. Oppenheim, Quarterly Journal of Mathematics, (2), vol. 2 (1931), pp. 230–233.

‡ E. T. Bell, this Bulletin, vol. 38 (1932), pp. 263–268.

MECHANICAL INVARIANTS OF THE SWEEPING-OUT PROCESS*

BY C. H. DIX

In this paper we prove the following theorem.

THEOREM. *If a general bounded distribution of positive mass in a closed connected region R is swept out on a surface S entirely enclosing R in its interior, then the center of gravity and the principal axes are invariants for the sweeping-out transformation.*

Let the distribution be given by $\Phi(e)$, which means the mass associated with the point set e. Then the potential is

$$V(M) = \int_R \frac{1}{\overline{MP}}\, d\Phi(e_P).$$

The coordinates $\bar{x}$, $\bar{y}$, $\bar{z}$ of the center of gravity of the distribution are, respectively,

$$\frac{1}{\Phi(R)}\int_R x_P d\Phi(e_P), \qquad \frac{1}{\Phi(R)}\int_R y_P d\Phi(e_P), \qquad \frac{1}{\Phi(R)}\int_R z_P d\Phi(e_P).$$

We have the following lemma.

LEMMA. *If Φ is such that a density ρ exists and $\nabla^2 V = -4\pi\rho$ is satisfied everywhere, then the sweeping-out on a level surface Σ of V entirely including R leaves the center of gravity and the principal axes invariant.*

The surface Σ is formed by setting $V=\delta>0$. Let P_0 be the center of gravity of the distribution Φ. Let R_0 be the lower bound of the radii of all spheres containing R with center P_0. If M_0 is a large integer, P on Σ, and $\delta=\Phi(R)(M_0R_0)^{-1}$, then we shall have $R_0(M_0-1)\leqq\overline{PP}_0\leqq R_0(M_0+1)$. Hence Σ lies in the spherical shell whose center is the point P_0 and whose bounding radii are $R_0(M_0-1)$ and $R_0(M_0+1)$.

Let u be any function harmonic in a closed region v containing P_0 whose boundary is the level surface Σ. Then, applying Green's Theorem, we have

* Presented to the Society, June 20, 1934.

$$\int_\Sigma u \frac{\partial V}{\partial n} d\sigma - \int_\Sigma V \frac{\partial u}{\partial n} d\sigma = \int_v (V\nabla^2 u - u\nabla^2 V) d\tau = 4\pi \int_v u\rho d\tau.$$

Since V is a constant on Σ and u is harmonic inside Σ,

$$\int_\Sigma V \frac{\partial u}{\partial n} d\sigma = V \int_\Sigma \frac{\partial u}{\partial n} d\sigma = 0.$$

Hence

$$\int_\Sigma u \frac{1}{4\pi} \frac{\partial V}{\partial n} d\sigma = \int_v u\rho d\tau.$$

Let $u = x$. Then

$$\int_\Sigma x \frac{1}{4\pi} \frac{\partial V}{\partial n} d\sigma = \int_v x\rho d\tau = \bar{x}\Phi(R),$$

with similar relations for $\bar{y}$, $\bar{z}$, $\overline{xy}$, $\overline{yx}$, and $\overline{zx}$. The expression $(\partial V/\partial n)/(4\pi)$ is of course the surface density of the swept-out mass on Σ. So the lemma is proved.

The extension to a general distribution is made by taking the iterated volume average of the potential until the corresponding mass distribution is sufficiently smooth to give rise to a potential satisfying Poisson's equation.

The treatment of these average functions has been carried out by G. C. Evans in a forthcoming paper.* They are used to prove the fundamental theorem of F. Riesz on the mass associated with a sub- or super-harmonic function. Now assuming the Riesz Theorem, let $\{\rho_i(P)\}$ be the sequence of positive densities corresponding to the super-harmonic functions $\{V(r_i, r_i, r_i, r_i; M)\}$ which are the fourth volume averages of V over spheres of center M and radius r_i. For each of these density distributions the conditions of the hypothesis in the lemma are satisfied.

For a small value of δ selected as in the lemma, $V_i(M) = V(r_i, r_i, r_i, r_i; M)$ is constant with respect to i on Σ, since the spherical volume average of a harmonic function gives the same harmonic function. Hence the same level surface Σ may be used at each stage in the sequence. At each stage

$$\int_\Sigma u \frac{1}{4\pi} \frac{\partial V_i}{\partial n} d\sigma = \int_v u\rho_i d\tau = \int_\Sigma u \frac{1}{4\pi} \frac{\partial V}{\partial n} d\sigma = \int_v u d\Phi_i(e_P),$$

* G. C. Evans, *On potentials of positive mass*, Transactions of this Society, vol. 37 (1935), No. 2.

where

$$\Phi_i(e) = \int_e \rho_i d\tau.$$

The limit of the sequence of integrals we may denote by $\bar{u}\Phi(R)$. Since u is bounded and continuous in v, and the Φ_i are of uniformly bounded variation on v, we may apply the Helly-Bray* theorem to obtain the result

$$\bar{u}\Phi(R) = \int_v u d\Phi(e_P).$$

Hence the lemma is true if Φ is a general bounded positive distribution.

To prove the theorem consider a level surface Σ of V enclosing both R and S in its interior. Consider the following sweeping-out transformations: T_{RS} = sweeping-out of R on S, $T_{S\Sigma}$ = sweeping-out of S on Σ, and $T_{R\Sigma}$ = sweeping-out of R on Σ. Now $T_{R\Sigma} \equiv T_{S\Sigma}T_{RS}$. Furthermore, $T_{R\Sigma}$ and $T_{S\Sigma}$ leave the center of gravity invariant. Thus we have *center of gravity of distribution on R = center of gravity of distribution on Σ = center of gravity of distribution on S.* A similar argument handles the principal axes.

If we have a closed surface S' bounding a region R' for which the Green's function can be constructed, this Green's function is the potential of the negative unit mass that has been swept out from the pole on S' and the positive unit point mass at the pole. Concerning the distribution of this swept-out mass we may observe the following property which is a corollary of our theorem: *the swept-out point mass has its center of gravity at the pole and its principal axes of inertia are arbitrary.*

That the distributions arising from the sweeping-out of a point mass are not the only ones with indeterminate principal axes is immediate. Take for example four equal point masses at the vertices of a regular tetrahedron (or take a homogeneous cube). In the cube the three moments of inertia about axes through the center normal to the faces are all equal. The momental ellipsoid is therefore a sphere. The momental ellipsoid for the tetrahedron has the same form relative to the four lines

* H. E. Bray, Annals of Mathematics, (2), vol. 20 (1918), pp. 177–186; see p. 180.

through the center of gravity and the vertices and so is a sphere.

The statement of our main theorem can be given in more general form but our statement is chosen on account of its intuitive simplicity. The set R we may take as merely closed and bounded; S may be the frontier of a bounded domain, D, which contains R. Then the conclusion remains the same as we have stated it in the simpler case.

THE RICE INSTITUTE

A DECOMPOSITION THEOREM FOR CLOSED SETS*

BY G. T. WHYBURN

Let P be any local† topological property of a closed set such that if K is any compact closed set lying in a metric space, then the set of all non-P-points of K is either vacuous or such that its closure is of dimension >0. The following are examples of such properties: (i) local connectivity, (ii) regularity (Menger-Urysohn sense), (iii) rationality, (iv) being of dimension $<n$, (v) belonging to no continuum of convergence, (vi) belonging to no continuum of condensation. In fact, it will be noted that in each of these cases, every non-P-point of a compact set K lies in a non-degenerate continuum of non-P-points of K. We proceed to prove the following theorem.

THEOREM. *If N denotes the set of all non-P-points of a compact closed set K in a metric space and if K is decomposed upper semi-continuously‡ into the components of $\overline{N}$ and the points of $K-\overline{N}$, then every point of the hyperspace H is a P-point of H.*

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* Presented to the Society, October 27, 1934.

† For the purposes of the present paper we shall understand by a local property of a set K a point property P such that if some neighborhood of a point x in K has property P at x, then K has property P at x; and conversely, if K has property P at x, then any neighborhood of x in K also has property P at x. A point x of K will be called a P-point or a non-P-point of K according as K does or does not have property P at x.

‡ For the notions relating to upper semi-continuous decompositions and for a proof that our particular decomposition is upper semi-continuous, the reader is referred to R. L. Moore, *Foundations of Point Set Theory*, American Mathematical Society, Colloquium Publications, 1932, Chapter 5.

PROOF. Let T denote the continuous transformation of K into H associated with our decomposition,* that is, $T(K)=H$ and for each $x \epsilon H$, $T^{-1}(x)$ is either a component of $\overline{N}$ or a point of $K-\overline{N}$. Then clearly $T(\overline{N})$ is a set of dimension 0. Thus if we suppose, contrary to our theorem, that H has at least one non-P-point, it follows by the condition on P that the set of all non-P-points of H can not be contained in $T(\overline{N})$. Hence there exists at least one point x in $K-\overline{N}$ such that $T(x)=y$ is a non-P-point of H. Let U be a neighborhood of x in K such that $U \cdot \overline{N}=0$. Then on U, T is a homeomorphism and $T(U)$ is a neighborhood of y in H. Since P is a local property and x is a P-point of K, x is also a P-point of U. Thus y is a P-point of $T(U)$ as U and $T(U)$ are homeomorphic; but then y must be a P-point also of H, because $T(U)$ is a neighborhood of y in H. Thus our supposition that not every point of H is a P-point leads to a contradiction and our theorem is proved.

COROLLARY. *If K is a compact metric continuum, then, choosing the properties* (i)–(vi) *above in turn, we find that the hyperspace of the decomposition is, respectively,* (i) *a locally connected continuum,* (ii) *a regular curve,* (iii) *a rational curve,* (iv) *a continuum of dimension $<n$,* (v) *an hereditarily locally connected continuum, and* (vi) *a continuum having no continuum of condensation.*

Part (i) of this corollary will be recognized at once as the well known result of R. L. Moore† to the effect that any continuum is a continuous curve with respect to its prime parts, for clearly the decomposition in this case is identically the decomposition of K into its prime parts.

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* See Alexandroff, Mathematische Annalen, vol. 96 (1926), p. 555; and Kuratowski, Fundamenta Mathematicae, vol. 11 (1928), p. 169.

† See Mathematische Zeitschrift, vol. 22 (1925), p. 308. For the notion of a prime part of a continuum together with a similar result for irreducible continua, see Hahn, Sitzungsberichte der Akademie der Wissenschaften in Wien, Mathematisch-Naturwissenschaftliche Klasse, vol. 130 (1921), pp. 217–250.

THE WEB OF QUADRIC HYPERSURFACES IN r DIMENSIONS*

BY T. R. HOLLCROFT

1. *Introduction.* Webs of quadric surfaces were defined and partially investigated in 1862 by de Jonquières.† Later, quadric webs were treated by many mathematicians, among whom were Cremona, Reye, Steiner, and Sturm.‡ Still later, the involution defined by a web of quadrics was studied by Snyder and Sharpe.§ There has been no treatment of webs in a space of higher dimension than three. The purpose of this paper is to derive the properties of a web of algebraic, quadric hypersurfaces in r dimensions.

2. *The Web.* The equation of a web of quadric hypersurfaces in S_r is

$$\sum \lambda_i f_i = 0, \qquad (i = 1, 2, 3, 4),$$

in which the f_i are entirely general quadratic functions of $x_1, x_2, \cdots, x_{r+1}$. The web has a basis manifold M_{r-4}^{16} of order 16 and dimension $r-4$. A doubly infinite non-linear system of quadric hypersurfaces of the web exists such that each hyperquadric of the system has a hypernode, that is, is a quadric hypercone. Also, among the hyperquadrics of the web, there is a doubly infinite non-linear system of pencils of hyperquadrics such that all the hyperquadrics of any one pencil have simple contact at one point. The locus of all such contacts as well as the locus of the hypernodes or vertices of the quadric hypercones of the web, is a surface J, the jacobian of the web. The characteristics of J will be found in §3.

The web also contains two singly infinite non-linear systems of pencils of hyperquadrics such that all the hyperquadrics in

* Presented to the Society, March 30, 1934.

† Journal de Mathématiques, (2), vol. 7 (1862), p. 412.

‡ Pascal, Repertorium der höheren Mathematik, vol. II_2 (1922), pp. 629–631. Encyklopädie der Mathematischen Wissenschaften, vol. III_2, second half, pp. 250–254.

§ Virgil Snyder and F. R. Sharpe, *Space involutions defined by a web of quadrics*, Transactions of this Society, vol. 19 (1918), pp. 275–290.

any one pencil have respectively two simple contacts at two points and one stationary contact at one point. The loci of the pairs of contacts and the stationary contacts are, respectively, two curves on J. Likewise, the web contains three finite systems of pencils of hyperquadrics such that in any one pencil the hyperquadrics have respectively three simple contacts, one simple and one stationary contact, one four-point contact. These contacts of higher order occur at points of the two curves of J. The orders of these curves and the number of hyperquadrics in each of the three finite systems are obtained in §6.

Since the equation of a quadric hypersurface in S_r contains $r(r+3)/2$ essential constants and since a hyperplane involves r conditions, $r(r+3)/2-2r$ conditions are necessary for a hyperquadric to degenerate into two hyperplanes. Since in the web there are but three conditions at our disposal, for $r(r+3)/2-2r>3$, when $r>3$, the web contains no pairs of hyperplanes. For $r=3$, however, as is well known, the web of quadric surfaces contains a finite number, ten, of pairs of planes.

The postulation of a line on a hypersurface of order n in S_r is $n+1$, which equals three for $n=2$. The three conditions that a hyperquadric contain a given line are the conditions that the hyperquadric contain three points of the line, which are linear in the coefficients of the hyperquadric. Therefore one and only one hyperquadric of the web is determined by a *given* line of S_r.

In determining the singular hyperquadrics of a web, however, we are concerned with the invariant postulation* I of the element on the hyperquadric. The formula for the invariant postulation is $I=P-q$, in which P is the postulation of the element on the hyperquadric and q is the number of conditions necessary to determine the position of the given element in S_r. For example, in the case of a hypernode on a hyperquadric, we have $P=r+1$, $q=r$, so that $I=1$. Hence the web of hyperquadrics contains ∞^2 hyperquadrics with a node, that is, ∞^2 quadric hypercones of the first species. Since a line is determined in S_r by $2(r-1)$ conditions, for a line on a hyperquadric in S_r, we have $I=3-2(r-1)=-2r+5$, that is, a hyperquadric of S_r contains ∞^{2r-5} lines.

* T. R. Hollcroft, *Invariant postulation*, this Bulletin, vol. 36 (1930), pp. 421–426.

The postulation P of a double line on a hyperquadric of S_r is $2r+1$. Hence, for a double line on a hyperquadric, $I=2r+1-2(r-1)=3$. The web of hyperquadrics, therefore, contains a finite number of hyperquadrics with a double line, that is, quadric hypercones of the second species. Such a hyperquadric of S_r can be projected from the double line into a general hyperquadric of S_{r-2}.* The number of quadric hypercones of the second species contained in the web is given in §5.

3. *The Jacobian.* The jacobian J is the surface that is common to the $r-2$ independent quartic hypersurfaces of the matrix $\|\partial f_i/\partial x_j\|$, $(i=1, 2, 3, 4; j=1, 2, \cdots, r+1)$; it is a surface of order n_0, class of plane section a_0, genus D_0. It contains l_0 lines, and has an apparent double curve of order b_0, which is of class q_0, and has t_0 apparent triple points, which are also apparent triple points of J. We find for J:

$$n_0 = \frac{1}{6}r(r^2-1), \qquad a_0 = \frac{1}{4}r(r-1)(r^2-r+2),$$

$$D_0 = \frac{1}{6}r(r-1)(r-2), \qquad l_0 = \frac{1}{6}r(r+1)(r+2),$$

$$b_0 = \frac{1}{72}r(r-1)(r-3)(r+4)(r^2+2),$$

$$2^3\cdot 3^4(q_0+4t_0) = (r-3)(2r^8+6r^7-15r^6-27r^5-129r^4 + 171r^3-236r^2+12r-432).$$

For $r=4$, we have $t_0=14$, $q_0=120$. The degrees of the expressions in r for q_0 and t_0 are 6 and 9, respectively. Sufficient data were not at hand to determine them separately for any r.

The jacobian surface J has no points in common with the basis manifold M_{r-4}^{16} of the web.

4. *The Associated Transformation and its Branch-Point Surface.* The transformation $\rho y_i=f_i$, $(i=1, 2, 3, 4)$, associates the hyperquadrics of the web uniquely with the planes of a three-space (y). The web contains ∞^4 pencils of hyperquadrics, images of the ∞^4 axial pencils of (y). The image of a line of (y) is the basis manifold M_{r-2}^4 of the corresponding pencil of hyper-

* E. Bertini, *Introduzione alla Geometria Proiettiva degli Iperspazi*, 1907, pp. 120–121.

quadrics. The manifold $M_r^{4}{}_{-2}$ is of order 4 and class $m_i=4(i+1)$, $(i=1, \cdots, r-2)$. The web contains ∞^3 nets of hyperquadrics, images of the ∞^3 bundles of planes of (y). The image of a point of (y) is the basis manifold $M_r^{8}{}_{-3}$ of the corresponding net.

The planes of (y) that correspond to quadric hypercones of the first species of the web envelop a surface L in (y) called the branch-point surface. The surfaces L and J are in (1, 1) correspondence. The order N of L is the number of intersections of J and two hyperquadrics of the web, so that $N=2r(r^2-1)/3$.

To obtain the characteristics of L, we must find the characteristics of the branch-point curve L_1 associated with a net of hyperquadrics in S_r. These characteristics have been derived in a former paper.* Setting $i=r$ and $n=2$ in the formulas, page 265, there results for the order n_1, class m_1, genus p_1; numbers of nodes δ_1, cusps κ_1, bitangents τ_1, inflections ι_1 of the curve L_1:

$$n_1 = r(r+1), \qquad m_1 = r+1, \qquad p_1 = \frac{1}{2}(r+1)(r-2)+1,$$

$$\delta_1 = \frac{1}{2}(r^2-1)(r-2)(r+4), \qquad \kappa_1 = 3(r^2-1), \qquad \tau_1 = \iota_1 = 0.$$

As has been proved in a former paper,† these are the characteristics of a plane section of the tangent cone of L.

The characteristics of L will be represented by these symbols:

$N[n']$ order [class];

$a[a']$ order of tangent cone [class of plane section];

$\kappa'[\delta']$ number of inflections [bitangents] of plane section;

$\kappa[\delta]$ number of cuspidal [nodal] lines of tangent cone;

$b[c]$ order of nodal [cuspidal] curve;

$b'[c']$ class of bitangential [spinodal] developable;

$q[r']$ class of nodal [cuspidal] curve;

$\beta[\gamma](i)$ number of intersections of nodal and cuspidal curves which are cusps on cuspidal [nodal] (neither) curve;

$\beta'[\gamma'](i')$ number of common planes of bitangential and spinodal developables which are stationary on the spinodal [bitangential] (neither) developable;

$t[t']$ number of triple points [planes] of nodal curve [bitangential developable];

* T. R. Hollcroft, *Nets of manifolds in i dimensions*, Annali di Matematica, (4), vol. 5 (1927–28), pp. 261–267.

† T. R. Hollcroft, Transactions of this Society, vol. 35 (1933), p. 859.

$\rho'[\sigma']$ order of bitangential [spinodal] curve;
$\rho[\sigma]$ class of nodal [cuspidal] developable;
C' number of conic tropes;
D genus.

Since the characteristics of the tangent cone are known and since from $b'=c'=0$, we obtain $\sigma'=\rho'=\beta'=\gamma'=\iota'=0$; the remaining characteristics of L are computed by means of the Cayley-Zeuthen equations relating the characteristics of reciprocal surfaces. There results:

$$N=\frac{2}{3}r(r^2-1), \qquad a=a'=r(r+1), \qquad n'=r+1,$$

$$\delta=\frac{1}{2}(r^2-1)(r-2)(r+4), \qquad \kappa=3(r^2-1),$$

$$\delta'=\frac{1}{6}r(r+1)(3r^2-8r+8), \qquad \kappa'=r(r^2-1),$$

$$b=\frac{1}{18}r(r+1)[4r(r-1)(r^2-1)-87r+159],$$

$$c=3r(r+1)(r-2), \qquad \sigma=4(r^2-1),$$

$$\rho=\frac{1}{3}(r+1)(2r^4-2r^2-39r+33),$$

$$\beta=2(r+1)(9r^2-28r+13), \qquad i=i'=0,$$

$$\gamma=2(r+1)[r^2(r^2-1)(r-2)-39r^2+114r-48],$$

$$t=\frac{1}{81}(r+1)(4r^8-4r^7-8r^6-253r^5+481r^4+257r^3+5130r^2-15570r+6075),$$

$$q=\frac{1}{3}(r+1)(2r^4-29r^2+45r-6),$$

$$r'=(r+1)(5r^2-10r+2), \qquad C'=\frac{1}{6}r(r+1)(r+2),$$

$$D=\frac{1}{6}r(r-1)(r-2).$$

The image of L is a hypersurface R in (x) of order $4r(r^2-1)/3$ which contains the jacobian surface J as a double surface.

5. *The Quadric Hypercones of the Second Species.* To the $r(r+1)(r+2)/6$ conic tropes of L correspond uniquely $r(r+1)(r+2)/6$ lines of J. To the plane which has contact with

L along a conic, forming a conic trope, corresponds a hyperquadric with the corresponding line of J as a double line, that is, a quadric hypercone of the second species. The web of hyperquadrics, therefore, contains $r(r+1)(r+2)/6$ quadric hypercones of the second species. The contact conic on L and the corresponding line on J are each the image of the other. For $r=3$, the $r(r+1)(r+2)/6$ quadric hypercones of the second species are ten pairs of planes.

6. *Loci of Contacts.* As has been stated, J is the locus of simple contacts of hyperquadrics of the web; J also contains all contacts of higher order.

To the cuspidal curve c of L corresponds a binodal manifold c_{r-2} of R of dimension $r-2$ and order $12r(r+1)(r-2)$. This manifold c_{r-2} and J both lie on R and intersect in a curve c_1 of order $3r(r+1)(r-2)/2$. The curve c_1 is a contact curve of J and c_{r-2}. The points of c and c_1 are in (1, 1) correspondence. The curve c_1 is the locus of the points of stationary contact of pencils of hyperquadrics of the web.

To the nodal curve b of L corresponds a nodal manifold b_{r-2} of R of dimensions $r-2$ and order

$$2r(r+1)[4r(r-1)(r^2-1)-87r+159]/9.$$

The surfaces J and b_{r-2} intersect in a curve b_1 whose order equals that of b. The image of b_1 is b counted twice. The points of b and b_1 are in (1, 2) correspondence. The curve b_1 is the locus of pairs of contacts of pencils of hyperquadrics of the web.

If P is a point γ of L, at which b and c intersect and b has a cusp, the image of P on J consists of two points, P_1 at an intersection of b_1 and c_1, and P_2 on b_1 only. The hyperquadrics of a pencil of the web have stationary contact at P_1 and simple contact at P_2. Since L has γ such points, the number of pencils of hyperquadrics of the web that have one stationary and one simple contact is the value of γ given in §4.

If P is a point β of L at which b and c intersect and c has a cusp, the image of P on J is a point P_0 which lies at a contact of b_1 and c_1. The hyperquadrics of a pencil of the web have 4-point contact at P_0. Since L has β such points, the number of pencils of hyperquadrics of the web that have 4-point contact is the value of β given in §4.

If P is a triple point t of b on L, P corresponds to three points of b_1. At these points, the hyperquadrics of three pencils, images of the three tangents to b at P, have simple contact. Hence the number of pencils of hyperquadrics of the web that have three simple contacts is $3t$, in which t has the value given in §4.

7. *Basis Elements.* As stated in §2, the web has a basis manifold of order 16 and dimension $r-4$. The web may have special basis elements which are equivalent to a part of the basis manifold M^{16}_{r-4}. A web of proper hyperquadrics can not have multiple basis elements.

In S_3, a web of quadrics with basis points is a special web. For $r=4$ the general web has 16 points, and for $r>4$ it has an infinite number contained in M^{16}_{r-4}.

For $r \leqq 4$, webs with basis lines are highly special. In S_5 a basis line becomes part of the basis curve and for $r \geqq 6$, the basis line lies on M^{16}_{r-4}. Similarly, for planes and elements of higher dimension. Webs of hyperquadrics may also have basis curves, surfaces, etc., of given order forming a part of or lying on M^{16}_{r-4}.

The basis elements $S_0, S_1, S_2, \cdots, S_\alpha$ have the respective postulation $1, 3, 6, \cdots, (\alpha+1)(\alpha+2)/2$ on a hyperquadric in S_r.

Since a web must contain three and only three arbitrary parameters, in establishing a web of hyperquadrics in S_r there are $r(r+3)/2-3$ constants at our disposal. For $r=3$, this number is 6, that is, a web may have a maximum of 6 basis points or 2 basis lines. For $r=4$, this number is 11, that is, a web in S_4 may have 1, 2 or 3 basis lines in which cases it has also 11, 6, 1 basis points, respectively. In S_r, finally, the maximum number of basis lines in a web of hyperquadrics is the largest integer contained in $(r^2+3r-6)/6$.

Similarly, a web of quadric hypersurfaces in S_4 may have one and only one basis plane, in which case it has in addition 5 basis points. The maximum number of basis planes in a web of quadric hypersurfaces in S_r is the largest integer contained in $(r^2+3r-6)/12$.

Finally, the maximum number of basis S_α's that can be assigned to a web of quadric hypersurfaces in S_r is the largest integer contained in $(r^2+3r-6)/[(\alpha+1)(\alpha+2)]$.

WELLS COLLEGE

A FACTORIZATION THEORY FOR POLYNOMIALS IN x AND IN FUNCTIONS $e^{\alpha x}$

BY L. A. MACCOLL

1. *Introduction.* In this note we consider the problem of determining all representations of a function of the form

$$f(x) = \sum_{n=0}^{N} \Phi_n(x) e^{\alpha_n x}, \tag{1}$$

where the Φ's are polynomials and the α's are constants, as a product of functions of the same form. The case in which the Φ's are constants has been discussed by J. F. Ritt.* As would be expected, the solution of the general problem possesses some features that are rather different from those appearing in the special case.

It is assumed, of course, that no one of the Φ's is identically zero, and that if $N>0$, no two of the α's are equal. The case of chief interest is that in which $N>0$ and in which the Φ's have no common zero. The discussion will be confined to this case. We select those of the α's for which the real parts are least, and of the constants so selected (if there be more than one) we select the one for which the coefficient of $(-1)^{1/2}$ is least. Let the constant so selected be denoted by α_0. We assume that $\alpha_0=0$. The class of functions of the form (1) satisfying the conditions stated in this paragraph will be called C.

If $f(x)$, $f_1(x)$, $\cdots$, $f_s(x)$ are all of the form (1), and if $f(x)=f_1(x)\cdots f_s(x)$, we shall say that $f(x)$ is divisible by each of the functions $f_i(x)$, and each of the latter functions will be called a factor of $f(x)$. A function which is divisible only by itself and by functions of the form $Ae^{\alpha x}$, where A and α are constants, will be called irreducible.

2. *Reduction to a Problem Concerning Polynomials.* Monomial factors of $f(x)$ are, in a certain sense, trivial. Henceforth we consider only factors having at least two terms. These factors may be taken as belonging to the class C.

Suppose that the function

* J. F. Ritt, *A factorization theory for functions* $\sum_{i=1}^{n} a_i e^{\alpha_i x}$, Transactions of this Society, vol. 29 (1927), pp. 584–596.

$$g(x) = \sum_{p=0}^{P} \Psi_p(x) e^{\beta_p x}$$

is a factor of $f(x)$. Ritt has shown in effect that each β is a linear combination of the α's with rational coefficients. He has also shown the existence of a set of numbers $\rho_1, \cdots, \rho_q$, which are linearly independent with respect to rational coefficients, and which are such that each α is a linear combination of the ρ's with non-negative integral coefficients. Each β is a linear combination of the ρ's with non-negative rational coefficients.

Consider a representation of $f(x)$ as a product of factors,

$$f(x) = \sum_{n=0}^{N} \Phi_n(x) e^{\alpha_n x} = \prod_{s=1}^{S} \sum_{p=0}^{P_s} \Psi_{s,p}(x) e^{\beta_{s,p} x}. \tag{2}$$

The α's and β's in this equation are understood to be expressed in terms of the ρ's as explained above. In each Φ and Ψ in (2) we replace x by the indeterminate y_0, and we replace each function $\exp(k\rho_i x)$, where k is a non-negative rational number, by y_i^k. Equation (2) is thus replaced by a relation in the y's which is easily seen to be an identity. Each of the indeterminates $y_1, \cdots, y_q$ can be replaced by a positive integral power of itself in such a way that the right-hand member of the relation in the y's obtained from (2) becomes a polynomial in the y's. The relation thus obtained is an identity.

We now have a method for factoring $f(x)$. First we express the α's as linear combinations, with non-negative integral coefficients, of numbers $\rho_1, \cdots, \rho_q$ which are linearly independent with respect to rational coefficients. In each of the polynomials Φ_n we replace x by y_0, and we replace $\exp(\rho_i x)$ by y_i, thus obtaining a polynomial $Q(y_0, y_1, \cdots, y_q)$. In this polynomial we replace the indeterminates y_i, $(i>0)$, in all possible ways by positive integral powers of themselves, thus obtaining a family of polynomials $Q(y_0, y_1^{i_1}, \cdots, y_q^{i_q})$. To each resolution of each of these polynomials into factors there corresponds a factorization of $f(x)$. All factorizations of $f(x)$ are obtained in this way.

3. *The Problem Concerning Polynomials.* Because of the conditions that we have imposed on $f(x)$, the polynomial Q has no non-constant monomial factor. Let $Q_1(y_0, y_1, \cdots, y_r)$ be an irreducible factor of Q. We shall consider the question: For which

positive integers t_i is the polynomial $Q_1(y_0, y_1^{t_1}, \cdots, y_\nu^{t_\nu})$ reducible? This is a modification of a problem studied by Ritt in his paper, and later by E. Gourin, who simplified the proofs and obtained stronger results.* The modification lies in the fact that, whereas in the problem of Ritt and Gourin all of the variables enjoy the same status, in our problem we may have one variable, y_0, that is exceptional in that it is not replaceable by a power of itself.

If Q_1 is independent of y_0, we have the case considered by Gourin. Henceforth we assume that Q_1 depends on y_0.

Suppose that Q_1 has at least three terms. By Gourin's theory, if there exists a set of positive integral t's such that the polynomial $Q_1(y_0^{t_0}, y_1^{t_1}, \cdots, y_\nu^{t_\nu})$ is reducible, there exists one and only one finite aggregate of sets of positive integers

$$t_{10}, t_{11}, \cdots, t_{1\nu}; \cdots; t_{M0}, t_{M1}, \cdots, t_{M\nu} \tag{3}$$

having the following properties. (1) For each m, the polynomial $Q_1(y_0^{t_{m0}}, \cdots, y_\nu^{t_{m\nu}})$ is reducible. (2) If $Q_1(y_0^{t_0}, \cdots, y_\nu^{t_\nu})$ is reducible, there exists in the aggregate (3) one and only one set, say $t_{j0}, \cdots, t_{j\nu}$, such that each t_i is an integral multiple of t_{ji}, say $t_i = \delta_i t_{ji}$, and the irreducible factors of $Q_1(y_0^{t_0}, \cdots, y_\nu^{t_\nu})$ are obtained by replacing each y_i by $y_i^{\delta_i}$ in the irreducible factors of $Q_1(y_0^{t_{j0}}, \cdots, y_\nu^{t_{j\nu}})$.

If no one of the numbers t_{m0} is unity, there exists no set of t's such that $Q_1(y_0, y_1^{t_1}, \cdots, y_\nu^{t_\nu})$ is reducible. Now suppose that $t_{10} = t_{20} = \cdots = t_{\mu 0} = 1$, and that $t_{m0} \neq 1$ for $m > \mu$. Then the sets

$$t_{11}, \cdots, t_{1\nu}; \cdots; t_{\mu 1}, \cdots, t_{\mu\nu} \tag{4}$$

have the following properties. (1) For each m, $1 \leqq m \leqq \mu$, $Q_1(y_0, y_1^{t_{m1}}, \cdots, y_\nu^{t_{m\nu}})$ is reducible. (2) If $Q_1(y_0, y_1^{t_1}, \cdots, y_\nu^{t_\nu})$ is reducible, there exists in (4) one and only one set, say $t_{j1}, \cdots, t_{j\nu}$, such that each t_i is an integral multiple of t_{ji}, say $t_i = \delta_i t_{ji}$, and the irreducible factors of $Q_1(y_0, y_1^{t_1}, \cdots, y_\nu^{t_\nu})$ are obtained by replacing $y_1, \cdots, y_\nu$ by $y_1^{\delta_1}, \cdots, y_\nu^{\delta_\nu}$, respectively, in the irreducible factors of $Q_1(y_0, y_1^{t_{j1}}, \cdots, y_\nu^{t_{j\nu}})$. The

* E. Gourin, *On irreducible polynomials in several variables which become reducible when the variables are replaced by powers of themselves*, Transactions of this Society, vol. 32 (1930), pp. 485–501.

aggregate (4) is the only aggregate of sets having these two properties.

Now suppose that Q_1 has only two terms, so that, with a proper assignment of the subscripts, it is of the form

$$Q_1 = ay_0^{\lambda_0}y_1^{\lambda_1}\cdots y_r^{\lambda_r} + by_{r-1}^{\lambda_{r-1}}\cdots y_s^{\lambda_s},$$

where a and b are constants, and the λ's are positive integers. By Gourin's theory, the infinite system of sets of positive integers

$$(5) \qquad t_{20}, t_{21}, \cdots, t_{2r}; t_{30}, t_{31}, \cdots, t_{3r}; \cdots,$$

where $t_{ki}=k/d_{ki}$, d_{ki} being the greatest common divisor of λ_i and k, has the following properties. (1) For each $k>1$, the polynomial $Q_1(y_0^{t_{k0}}, \cdots, y_r^{t_{kr}})$ is reducible. (2) If $Q_1^{(t)}=Q_1(y_0^{t_0}, \cdots, y_r^{t_r})$ is reducible, there exists in (5) one and only one set, say $t_{j0}, \cdots, t_{jr}$, such that each t_i is an integral multiple of t_{ji}, say $t_i=\delta_i t_{ji}$, and the irreducible factors of $Q_1^{(t)}$ are obtained by replacing each y_i by $y_i^{\delta_i}$ in the irreducible factors of $Q_1(y_0^{t_{j0}}, \cdots, y_r^{t_{jr}})$. The system (5) is the only system of sets of integers having these two properties.

If no one of the numbers $t_{20}, t_{30}, \cdots$ is unity, there is no set of positive integral t's for which $Q_1(y_0, y_1^{t_1}, \cdots, y_r^{t_r})$ is reducible. Now suppose that $t_{m_1 0}=t_{m_2 0}=\cdots=1$, and that the remaining numbers of the set $t_{20}, t_{30}, \cdots$ are all different from unity. Then the system of sets

$$(6) \qquad t_{m_1 1}, \cdots, t_{m_1 r}; t_{m_2 1}, \cdots, t_{m_2 r}; \cdots$$

has the following properties. (1) For any set of (6) the polynomial $Q_1(y_0, y_1^{t_{m_h 1}}, \cdots, y_r^{t_{m_h r}})$ is reducible. (2) If for any positive integral t's the polynomial $Q_1(y_0, y_1^{t_1}, \cdots, y_r^{t_r})$ is reducible, there exists in (6) one and only one set, say $t_{m_j 1}, \cdots, t_{m_j r}$, such that each t_i is an integral multiple of $t_{m_j i}$, say $t_i=\delta_i t_{m_j i}$, and the irreducible factors of $Q_1(y_0, y_1^{t_1}, \cdots, y_r^{t_r})$ are obtained by replacing $y_1, \cdots, y_r$ by $y_1^{\delta_1}, \cdots, y_r^{\delta_r}$, respectively, in the irreducible factors of $Q_1(y_0, y_1^{t_{m_j 1}}, \cdots, y_r^{t_{m_j r}})$. The system (6) is the only system of sets of positive integers having these two properties.

The system (6) does not exist if $\lambda_0=1$; if $\lambda_0>1$, the system exists, the integers $m_1, m_2, \cdots$ being precisely the divisors of λ_0 that are greater than 1. If the system (6) exists, the number of

sets in it is finite. This is the outstanding novelty introduced by the presence of the exceptional variable y_0.

4. *The Factorization Theorem.* We are now ready to complete the solution of our problem. We begin by resolving Q into its irreducible factors. The irreducible factors which do not contain y_0 are treated as in Ritt's theory. Those which contain at least three terms give rise to irreducible factors of $f(x)$. Those which have only two terms give rise to a definite set of factors of the form $b_0+\sum_{i=1}^{m} b_i \exp(\beta_i x)$, where the b's are constants, the β's in each function have rational ratios to one another, and any two β's in different functions have an irrational ratio. Ritt calls these factors *simple functions*. A simple function has an infinite set of factors.

Consider an irreducible factor of Q, say Q_1, which contains y_0. We have seen that either there are no positive integral t's for which $Q_1(y_0, y_1^{t_1}, \cdots, y_\nu^{t_\nu})$ is reducible, or there exists one and only one finite aggregate of sets of positive integers

$$t_{m1}, \cdots, t_{m\nu}, \qquad (m = 1, \cdots, M),$$

which have the following properties. (1) For each value of m, $Q_1(y_0, y_1^{t_{m1}}, \cdots, y_\nu^{t_{m\nu}})$ is reducible. (2) If $Q_1(y_0, y_1^{t_1}, \cdots, y_\nu^{t_\nu})$ is reducible, there is one and only one value of m, say μ, such that each t_i is an integral multiple of $t_{\mu i}$, say $t_i = \delta_i t_{\mu i}$, and the irreducible factors of $Q_1(y_0, y_1^{t_1}, \cdots, y_\nu^{t_\nu})$ are obtained by replacing $y_1, \cdots, y_\nu$ by $y_1^{\delta_1}, \cdots, y_\nu^{\delta_\nu}$, respectively, in the irreducible factors of $Q_1(y_0, y_1^{t_{\mu 1}}, \cdots, y_\nu^{t_{\mu\nu}})$.

If no set of t's exists such that $Q_1(y_0, y_1^{t_1}, \cdots, y_\nu^{t_\nu})$ is reducible, the function $Q_1(x, e^{\rho_1 x}, \cdots, e^{\rho_\nu x})$ is an irreducible factor of $f(x)$.

If a set of t's exists such that $Q_1(y_0, y_1^{t_1}, \cdots, y_\nu^{t_\nu})$ is reducible, we have a relation of the form (in the notation used above)

$$Q_1(y_0, y_1^{t_{m1}}, \cdots, y_\nu^{t_{m\nu}}) = \prod_{k=1}^{K_m} Q_{1k}(y_0, y_1, \cdots, y_\nu),$$

where the Q's in the second member are irreducible. Let the value of m be selected so that K_m has its maximum value. Then each of the functions

$$(7) \qquad Q_{1k}(x, e^{\rho_1 x/t_{m1}}, \cdots, e^{\rho_\nu x/t_{m\nu}}), \qquad (k = 1, \cdots, K_m),$$

is irreducible. In fact if this were not so, there would exist a

set of positive integers δ_i such that $Q_1(y_0, y_1^{\delta_1}, \cdots, y_\nu^{\delta_\nu})$ would be resolvable into more than K_m factors, which is not the case. Each of the functions (7) is a factor of $f(x)$.

When we multiply together the simple functions coming from the irreducible binomial factors of Q which do not involve y_0 and the irreducible functions coming from the remaining irreducible factors of Q, we have a resolution of $f(x)$ into factors belonging to the class C. It is easily seen that this factorization is unique. Thus we have the following theorem.

THEOREM. *A function $f(x)$ belonging to the class C can be expressed in one and only one way as a product*

$$f(x) = I_1(x) \cdots I_m(x) S_1(x) \cdots S_n(x),$$

where each factor belongs to C, the I's are irreducible functions, and the S's are simple functions, $b_0 + \sum b_i \exp(\beta_i x)$, such that the ratio of any two β's in different functions is irrational.

BELL TELEPHONE LABORATORIES

THE NUMBER OF TRISECANTS OF A SPACE CURVE OF ORDER m WHICH MEET AN i-FOLD SECANT*

BY L. A. DYE

The number of trisecants of a space curve C_m, of order m, which meet a general line was determined by Zeuthen,† but if the line happens to be an i-fold secant, $i > 2$, it lies on the ruled surface of trisecants and the formula fails. In algebraic geometry some extension of Zeuthen's work to cover this neglected case is often necessary, so by means of a correspondence we show that the number of trisecants of a C_m which meet an i-fold secant l is

$$(m-2)[h - m(m-1)/6] - i(h - m + 2) + i(i-1)(i-2)/6,$$

where h is the number of apparent double points of C_m.

In the plane determined by l and one of the $h' = h - i(i-1)/2$

* Presented to the Society, October 27, 1934.

† H. G. Zeuthen, *Sur les singularités des courbes gauches*, Annali di Matematica, (2), vol. 3 (1869), pp. 175–217.

bisecants of C_m through any point λ on l, let A, B be the points of C_m on the bisecant and let C_j, $[j=1, 2, \cdots, (m-i-2)]$, denote the points of C_m not on l or the bisecant. The lines AC_j and BC_j determine $2(m-i-2)$ points μ on l, and when all of the h' bisecants through the point λ are considered there are $2(m-i-2)h'$ points μ determined by each point λ. Since the relationship between the points λ and μ is symmetrical, there exists a $[2(m-i-2)h',\ 2(m-i-2)h']$ correspondence. The $4(m-i-2)h'$ coincidences of this correspondence fall into three classes.

1. In the plane determined by l and a trisecant of C_m meeting l let A, B, C be the points of C_m on the trisecant. If the line is thought of as AB, then AC and BC each account for a coincidence. Similarly the line may be taken as AC or BC, so that 6 coincidences arise from each of the x trisecants of C_m meeting l.

2. Since there are $r=m(m-1)-2h$ tangents to C_m meeting an arbitrary line, there are $r'=r-2i$ tangents meeting l. In the plane of one of these tangents and l, let $A\equiv B$ be the point of tangency and C_j any one of the $m-i-2$ residual intersections of C_m not on l. For each line AC_j there arises one coincidence due to BC_j, hence there are $(m-i-2)r'$ coincidences due to the tangent lines meeting l.

3. In the plane determined by l and a tangent to C_m at one of the i intersections with l, call the point of tangency $A\equiv B$. Join A to one of the $m-i-1$ residual intersections C_j of C_m, then the lines joining B to the $m-i-2$ remaining points C_k determine $m-i-2$ coincidences. Since there are $m-i-1$ choices for C_j, and i points on l, there are $i(m-i-1)\ (m-i-2)$ coincidences accounted for in this case.

We now solve the equation

$$4(m-i-2)h' = 6x + (m-i-2)r' + i(m-i-1)(m-i-2),$$

and obtain

$$x=(m-2)[h-m(m-1)/6]-i(h-m+2)+i(i-1)(i-2)/6.$$

CORNELL UNIVERSITY

DEGREE OF APPROXIMATION BY POLYNOMIALS TO CONTINUOUS FUNCTIONS*

BY W. E. SEWELL

1. *Introduction.*† This paper is primarily concerned with functions analytic in a given region and continuous in the corresponding closed region; the region is in all cases bounded by an analytic Jordan curve with no double points. The existence of polynomials which converge uniformly to the function in the closed region has been established by Walsh‡ for more general regions, but it is frequently convenient to have more precise results in the study of functions on the boundary of the region. The method used here is essentially a reduction of the problem to the expansion of a function in a Fourier series and consequently the function is assumed to satisfy a Lipschitz or Hölder condition. An extension of the classical theory of Fourier series, identification of this expansion with the Taylor expansion on the circumference of the unit circle, and a study of the degree of approximation of Faber's polynomials belonging to the region form the basis of this investigation.

2. *Relation between the Fourier and Taylor Expansions.* The following theorem can be proved easily by separating the function into its real and imaginary parts and examining the coefficients.

THEOREM 1. *Let $F(x)$ be analytic in $|x|<1$, continuous in $|x|\leqq 1$. Then the Taylor development for $F(x)$ about $x=0$ is precisely the Fourier development for $F(x)$ on the circumference, $|x|=1$.*

If $F(x)$, besides being continuous in the closed circle, satisfies

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* Presented to the Society, December 28, 1934.

† This is a portion of a thesis written at Harvard University under the direction of Professor J. L. Walsh, and I am indebted to him for constant aid and encouragement.

‡ J. L. Walsh, Mathematische Annalen, vol. 96 (1926), pp. 430–436 and pp. 437–450.

a Hölder condition* of order α, $(0<\alpha\leqq 1)$, then the results on degree of approximation of Fourier series may be applied, where $F(x)$ is considered as a complex function of the real variable θ. It is necessary in this connection to show that $|F(x_1)-F(x_2)| < M|x_1-x_2|^\alpha$, $x_k=e^{i\theta_k}$, implies

$$|u(\theta_1)-u(\theta_2)| < M_1|\theta_1-\theta_2|^\alpha, \quad |v(\theta_1)-v(\theta_2)| < M_1|\theta_1-\theta_2|^\alpha,$$

where $F(e^{i\theta})=u(\theta)+iv(\theta)$, and this can be done without difficulty. *Also if* $|F^{(p)}(x_1)-F^{(p)}(x_2)| < M|x_1-x_2|^\alpha$, *where* p *is a positive integer or zero and* $F^{(p)}(x)$ *is the* p*th derivative (the zero-th derivative is the function itself), then the* p*th derivatives of* $u(\theta)$ *and* $v(\theta)$ *satisfy the corresponding conditions in* θ.

3. *Degree of Convergence of Fourier Series.*† A classical result in Fourier series is the following:

THEOREM 2. *If* $|f(x_1)-f(x_2)| \leqq M|x_1-x_2|$ *for all values of* x_1 *and* x_2, M *being a constant, then* $|f(x)-S_n(x)| \leqq (AM\log n)/n$, *where* A *is an absolute constant and* $S_n(x)$ *is the sum of the first* n *terms of the Fourier series for* $f(x)$.

This theorem in *real variables* can be extended‡ to include Hölder conditions of order α, $(0<\alpha<1)$. Now by an application of the results of §2 we get the following theorem.

THEOREM 3. *If* $F(x)$ *is analytic in* $|x|<1$, *continuous in* $|x|\leqq 1$, *and on the circumference* $|F(x_1)-F(x_2)| < M|x_1-x_2|$, *then*

$$|F(x)-P_n(x)| < (M_1\log n)/n, \qquad |x| \leqq 1,$$

where $P_n(x)$ *is the sum of the first* n *terms of the Taylor development of* $F(x)$ *about* $x=0$.

Finally, by further application of the results on Fourier series and the relation to the Taylor development, we get the following theorem.

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* That is, $|F(x_1)-F(x_2)| < M|x_1-x_2|^\alpha$; here the Hölder condition will be used to include the case where $\alpha=1$.

† For the results used here on real variable approximation see Dunham Jackson, *Theory of Approximation*, 1930.

‡ See Jackson, loc. cit., pp. 4 ff. The proof goes through with slight modification for this case.

THEOREM 4. *If $F(x)$ is analytic in $|x|<1$, continuous in $|x|\leqq 1$, and $|F^{(p)}(x_1)-F^{(p)}(x_2)|\leqq M|x_1-x_2|^{\alpha}$, $(0<\alpha\leqq 1)$, where p is a positive integer or zero, then*

$$|F(x)-P_n(x)|\leqq (M_1\log n)/n^{p+\alpha}, \qquad |x|\leqq 1.$$

4. *Faber's Polynomials.* Let C be an analytic Jordan curve in the x plane and let

$$(1)\quad x=\psi(t)=1/t+a_0+a_1t+a_2t^2+\cdots=1/t+\mathfrak{P}(t)$$

map the exterior of C on the region $|t|<r$ of the complex t plane, carrying $x=\infty$ into $t=0$. Due to the analyticity of C, the right-hand side of (1) converges for $|t|\leqq r'$, $r'>r$, and yields a uniquely inversible map. Denote $|t|=\rho<r'$ by K_ρ and let its image in the x plane be C_ρ; then $C_r=C$. Faber's polynomials* for the given region are defined as follows: $P_n(x)$ is the polynomial in x of degree n such that the coefficient of x^n is unity, and, as a function of t through (1), it has zero coefficients for the terms in $t^{-n+1}, t^{-n+2}, \cdots, t^{-1}, t^0$; hence $P_n(x)=1/t^n+t\mathfrak{P}_n(t)$, where $\mathfrak{P}_n(t)$ converges for $|t|\leqq r'$. Faber has proved that *any function analytic interior to C can be developed in a series, $\sum_{r=0}^{\infty}a_rP_r(x)$, which converges to the function at every interior point, and, conversely, every series which converges in the interior of C represents an analytic function.*

5. *Convergence to $F(x)$ in the Closed Region.* Now let $F(x)$ be an arbitrary function analytic in the region bounded by C_ρ, an arbitrary analytic Jordan curve, let the function be continuous in the corresponding closed region, and let $F(x)$ satisfy a Lipschitz condition (Hölder condition of order one) on C_ρ. Under these conditions the convergence of Faber's polynomials to $F(x)$ in the closed region will be established.

By Faber's results $F(x)=\sum_0^{\infty}a_rP_r(x)$, x within C_ρ. Since the curves are analytic, $X=\psi(\tau)$, where X is on C_ρ and τ is on K_ρ, is analytic and, consequently, $|F(X_1)-F(X_2)|\leqq A|X_1-X_2|$ implies

$$|F(\psi(\tau_1))-F(\psi(\tau_2))|\leqq A|\psi(\tau_1)-\psi(\tau_2)|,$$

where $X_1=\psi(\tau_1)$, $X_2=\psi(\tau_2)$. But since $\psi(t)$ is analytic on K_ρ, $|\psi(\tau_1)-\psi(\tau_2)|\leqq B|\tau_1-\tau_2|$, and hence

* G. Faber, Mathematische Annalen, vol. 57, pp. 389–408; Journal für Mathematik, 1920.

$$|F(\psi(\tau_1)) - F(\psi(\tau_2))| \leqq AB|\tau_1 - \tau_2| = C|\tau_1 - \tau_2|.$$

Thus $F(\psi(t))$ satisfies a Lipschitz condition on K_ρ. From above,

$$F(\psi(t)) = \sum_0^\infty a_\nu(1/t^\nu + t\mathfrak{P}_\nu(t)), \qquad (\rho < |t| < r'),$$

where $|t\mathfrak{P}_\nu(t)| < G/r'^\nu$, G fixed; $a_\nu = [1/(2\pi i)]\int_{K_\rho} F(\psi(\tau))\tau^{\nu-1}d\tau$. We may write $F(\psi(t))$ in the form

$$F(\psi(t)) = \sum_0^\infty a_\nu/t^\nu + \sum_1^\infty b_\nu t^\nu, \qquad (\rho < |t| < r'). \tag{2}$$

Since the right-hand side is analytic in $\rho < |t| < r'$, $\sum_1^\infty b_\nu t^\nu$ is analytic in $|t| < r'$. Now define $\phi(t) = -\sum_1^\infty b_\nu t^\nu + F(\psi(t))$. This function is analytic in $\rho < |t| < r'$, $F(\psi(t))$ satisfies a Lipschitz condition on $|t| = \rho$ and $-\sum_1^\infty b_\nu t^\nu$ is analytic in $|t| < r'$, and hence $\phi(t)$ satisfies a Lipschitz condition on $|t| = \rho$. From (2), $\phi(t) = \sum_0^\infty a_\nu/t^\nu$, $(\rho < |t| < r')$, and under the transformation $t = \rho/y$, $\phi(\rho/y) = \Phi(y)$, $\Phi(y)$ is represented by its Taylor development* in the ring $\rho/r' < |y| < 1$. The function $\Phi(y)$ may be defined throughout the unit circle by this convergent power series, and since $\Phi(y)$ is analytic throughout the unit circle, and satisfies a Lipschitz condition on the circumference, by Theorem 3 the Taylor development converges to $\Phi(y)$ on $|y| = 1$. This means that

$$-\sum_1^\infty b_\nu t^\nu + F(\psi(t)) = \sum_0^\infty a_\nu/t^\nu, \qquad (\rho \leqq |t| < r');$$

and hence

$$F(\psi(t)) = \sum_0^\infty a_\nu/t^\nu + \sum_1^\infty b_\nu t^\nu, \qquad (\rho \leqq |t| < r'),$$

which means *convergence in the closed region.*

By Theorem 3 we have $|\Phi(y) - \sum_0^n a_\nu y^\nu/\rho^\nu| < (M \log n)/n$ on and interior to $|y| = 1$. But $y = \rho/t$ and $\Phi(y) = \phi(\rho/y) = \phi(t) = F(\psi(t)) - \sum_1^\infty b_\nu t^\nu$, whence

$$\left| F(\psi(t)) - \sum_1^\infty b_\nu t^\nu - \sum_0^n a_\nu/t^\nu \right| < (M \log n)/n, \qquad (|t| = \rho),$$

* The original expression for a_ν under the transformation above serves to show that a_ν/ρ^ν is the νth Taylor coefficient of $\Phi(y)$.

which means that

$$\left| \sum_{n+1}^{\infty} a_\nu / t^\nu \right| < (M \log n)/n, \qquad (|t| = \rho).$$

But

$$\sum_{1}^{\infty} b_\nu t^\nu = \sum_{1}^{\infty} a_\nu t \mathfrak{P}_\nu(t) = \sum_{1}^{n} a_\nu t \mathfrak{P}_\nu(t) + \sum_{n+1}^{\infty} a_\nu t \mathfrak{P}_\nu(t),$$

and hence

$$\begin{aligned} F(\psi(t)) - \sum_{1}^{\infty} b_\nu t^\nu - \sum_{0}^{n} a_\nu / t^\nu &= F(\psi(t)) - \sum_{1}^{\infty} a_\nu t \mathfrak{P}_\nu(t) - \sum_{0}^{n} a_\nu / t^\nu \\ &= F(\psi(t)) - \sum_{1}^{n} a_\nu t \mathfrak{P}_\nu(t) - \sum_{0}^{n} a_\nu / t^\nu - \sum_{n+1}^{\infty} a_\nu t \mathfrak{P}_\nu(t) \\ &= F(\psi(t)) - \sum_{0}^{n} a_\nu (1/t^\nu + t \mathfrak{P}_\nu(t)) - \sum_{n+1}^{\infty} a_\nu t \mathfrak{P}_\nu(t). \end{aligned}$$

Thus

$$\begin{aligned} \left| F(\psi(t)) - \sum_{0}^{n} a_\nu (1/t^\nu + t \mathfrak{P}_\nu(t)) \right| &\leqq \left| \sum_{n+1}^{\infty} a_\nu / t^\nu \right| + \left| \sum_{n+1}^{\infty} a_\nu t \mathfrak{P}_\nu(t) \right| \\ &\leqq (M \log n)/n + \sum_{n+1}^{\infty} |a_\nu| \, |t \mathfrak{P}_\nu(t)|. \end{aligned} \tag{3}$$

But $|t \mathfrak{P}_\nu(t)| < G/r'^\nu$, G fixed, $\rho < r'$, $|t| < r'$, and

$$|a_\nu| = \left| \frac{1}{2\pi i} \int_{K_\rho} F(\psi(\tau)) \tau^{\nu-1} d\tau \right| \leqq 2\pi\rho M \rho^{\nu-1}/(2\pi) = M\rho^\nu.$$

Since $F(x)$ satisfies a Lipschitz condition on C_ρ, it is bounded there and thus $F(\psi(t))$ is bounded on K_ρ. Hence

$$\begin{aligned} \sum_{n+1}^{\infty} |a_\nu| \, |t \mathfrak{P}_\nu(t)| &\leqq \sum_{n+1}^{\infty} M\rho^\nu G / r'^\nu = \sum_{n+1}^{\infty} MG(\rho/r')^\nu \\ &= MG \sum_{n+1}^{\infty} (\rho/r')^\nu. \end{aligned} \tag{4}$$

But the remainder of a geometric series with quotient less than unity is of order less than $(\log n)/n$.* In consideration of this fact, an application of (4) to (3) yields

* This follows directly from the fact that nx^{n+1} for $|x| < 1$ is the nth term of a convergent series. We shall use the fact that this geometric remainder is less than $(\log n)/n^\alpha$, $(0 < \alpha < 1)$, in a later proof.

$$(5) \quad \left| F(\psi(t)) - \sum_0^n a_\nu (1/t^\nu + t\mathfrak{P}_\nu(t)) \right| \leqq (A' \log n)/n, \ (t \text{ on } K_\rho).$$

The constant A' can be adjusted so that this inequality holds for all n. Since $x=\psi(t)$ and $P_\nu(x)=1/t^\nu+t\mathfrak{P}_\nu(t)$, (4) becomes

$$\left| F(x) - \sum_0^n a_\nu P_\nu(x) \right| \leqq (A' \log n)/n.$$

The result can be stated as follows.

THEOREM 5. *If $F(x)$ is analytic interior to the analytic Jordan curve C_ρ, continuous in the closed region, and satisfies a Lipschitz condition on C_ρ, then*

$$\left| F(x) - \sum_0^n a_\nu P_\nu(x) \right| \leqq (A' \log n)/n,$$

x on or within C_ρ, where A' is a constant independent of x and n (A' depends on $F(x)$ and on C_ρ) and $P_\nu(x)$ is the Faber polynomial of degree ν belonging to the region.

6. *Extension to Hölder Conditions and Derivatives.* Suppose $F(x)$ satisfies on C_ρ a Hölder condition of order α, $(0<\alpha\leqq 1)$, with the remaining hypotheses unchanged. By examination of the above it is seen that the first modification will come in (2) and by Theorem 4 the inequality will be

$$\left| \Phi(y) - \sum_0^n a_\nu y^\nu/\rho^\nu \right| < (M \log n)/n^\alpha.$$

The geometric remainder can be handled as before,* and we get the following theorem.

THEOREM 6. *If $F(x)$ is analytic interior to C_ρ, continuous in the corresponding closed region, and if*

$$|F(X_1) - F(X_2)| \leqq M |X_1 - X_2|^\alpha, \qquad (0 < \alpha \leqq 1),$$

X_1, X_2 on C_ρ, then

$$\left| F(x) - \sum_0^n a_\nu P_\nu(x) \right| \leqq (A' \log n)/n^\alpha, \ x \text{ on or within } C_\rho.$$

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* See preceding footnote.

Now suppose, we have, instead of a Lipschitz condition on the function, $|F^{(p)}(X_1)-F^{(p)}(X_2)| \leqq M|X_1-X_2|^{\alpha}$, $(0<\alpha\leqq 1)$, p a positive integer. Let $F(x)=F(\psi(t))=f(t)$ and $t=\phi(x)$. Then $dt=\phi'(x)dx$, $dt/dx=\phi'(x)$. Since $F(x)$ has a pth derivative on C_ρ, $f(t)$ has a pth derivative on K_ρ. We have

$$F'(x) = \frac{dF}{dx} = \frac{df}{dt}\frac{dt}{dx} = f'(t)\phi'(t),$$

$$F''(x) = \frac{d}{dx}\frac{dF}{dx} = \frac{d}{dx}[f'(t)\phi'(x)] = f'(t)\phi''(x) + \phi'(x)f''(t)\phi'(x)$$

$$= f'(t)\phi''(x) + [\phi'(x)]^2 f''(t),$$

$$F'''(x) = \phi'''(x)f'(t) + \phi''(x)\phi'(x)f''(t)$$

$$+ 2[\phi'(x)][\phi''(x)]f''(t) + [\phi'(x)]^3 f'''(t), \text{ etc.}$$

Now, since $\phi(x)$ is analytic, all its derivatives satisfy Lipschitz conditions on C_ρ and hence with respect to t on K_ρ, and since $f(t)$ has a pth derivative on K_ρ its derivatives of all orders less than p satisfy Lipschitz conditions there, and hence it is seen that $f^{(p)}(t)$ satisfies the corresponding Hölder condition on K_ρ. The geometric remainder can be treated just as above, and thus we have the following theorem.

THEOREM 7. *Let $F(x)$ be analytic in the region bounded by the analytic Jordan curve C_ρ, continuous in the corresponding closed region, and further let*

$$|F^{(p)}(X_1) - F^{(p)}(X_2)| \leqq M|X_1 - X_2|^{\alpha}, \qquad (0 < \alpha \leqq 1),$$

X_1, X_2 on C_ρ, p a positive integer or zero; then

$$\left| F(x) - \sum_0^n a_\nu P_\nu(x) \right| \leqq (M_1 \log n)/n^{p+\alpha}, \text{ } x \text{ on or within } C_\rho.$$

This theorem includes all the preceding results as special cases.

HARVARD UNIVERSITY

A POLAR REPRESENTATION OF SINGULAR MATRICES

BY JOHN WILLIAMSON

Let $A=(a_{ij})$, $(i=1, 2, \cdots, m; j=1, 2, \cdots, n)$, be a matrix of m rows and n columns, whose elements a_{ij} are complex numbers. It has been shown† that, if $m=n$ and A is non-singular, $A=P_1U=UP_2$, where U is a unitary matrix, while P_1 and P_2 are positive definite hermitian matrices. Moreover in such a polar representation of A, as it has been called, the matrices P_1, P_2, and U are uniquely determined. We shall show that, if $m=n$ and the rank of A is $r<n$, $A=P_1U=UP_2$, where P_1 and P_2 are uniquely determined positive hermitian matrices of rank r and U is unitary but no longer unique. Any such representation of course is impossible if $m\neq n$, as by definition both hermitian and unitary matrices are square, but it will be shown that somewhat analogous results exist in this case as well.

As is customary we shall denote the conjugate transposed of A by $A^*=(a_{ij}^*)$, where $a_j^*=\bar{a}_{ji}$, the complex conjugate of a_{ji}. We shall use this notation, even if A is a vector, that is, a matrix of one row, so that in this case AA^* will simply denote the norm of the vector A. For the sake of brevity we shall use the notations E_j for the unit matrix of order j and $0_{i,j}$ for the zero matrix of i rows and j columns.

The matrix $N_1=AA^*$ is a square matrix of order m and the matrix $N_2=A^*A$ is a square matrix of order n, and since $N_1=N_1^*$ and $N_2=N_2^*$, both of these matrices are hermitian. Moreover, if the rank of A is r, the rank of N_1 is r and so is the rank of N_2. For, if K is the rth compound‡ of A, at least one element k_{ij} of K is different from zero. The element in the ith place of the leading diagonal of the product matrix KK^* is $\sum_t k_{it}\bar{k}_{it}$, which is a positive real number, since k_{ij} is not zero. Accordingly there is at least one r-rowed determinant of N_1

† L. Autonne, Bulletin de la Société Mathématique, vol. 30 (1902), pp. 121–134. A. Wintner and F. D. Murnaghan, *On a polar representation of non-singular matrices*, Proceedings of the National Academy of Sciences, vol. 17 (1931), pp. 676–678.

‡ Turnbull and Aitken, *The Theory of Canonical Matrices*, p. 27.

which is not zero, so that the rank of N_1 is at least r. Since the rank of N_1 cannot exceed the rank r of A, the rank of N_1 is exactly r. Similarly the rank of N_2 is r.

Since N_1 is a hermitian matrix of rank r, there exists an m-rowed unitary matrix X such that†

$$XN_1X^* = D = \begin{pmatrix} D_{11} & 0_{r,m-r} \\ 0_{m-r,r} & 0_{m-r,m-r} \end{pmatrix}, \tag{1}$$

where D_{11} is a diagonal matrix of order r. If $B = XA$, $B^* = A^*X^*$, so that

$$BB^* = XAA^*X^* = XN_1X^* = D. \tag{2}$$

If we denote the row vectors of B by b_i, $(i = 1, 2, \cdots, m)$, the column vectors of B^* are b_i^*, $(i = 1, 2, \cdots, m)$, and the element in the ith row and jth column of BB^* is $b_ib_j^*$. It therefore follows from (1) and (2) that $b_ib_i^* = 0$, $(i = r+1, \cdots, m)$. Hence B is a matrix whose last $m-r$ rows are zero so that

$$B = \begin{pmatrix} B_1 \\ 0_{m-r,n} \end{pmatrix}, \tag{3}$$

where B_1 is a matrix of r rows and n columns. By a similar argument applied to N_2 instead of to N_1, it can be shown that there exists an n-rowed unitary matrix Y such that

$$AY = (B_2 \quad 0_{m,n-r}), \tag{4}$$

where B_2 is a matrix of m rows and r columns. From (3) and (4) we deduce that

$$XAY = C = \begin{pmatrix} C_{11} & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{pmatrix}, \tag{5}$$

where C_{11} is an r-rowed square matrix, which is non-singular since the rank of A is r. Since $XN_1X^* = XAYY^*A^*X^* = CC^*$, it follows from (1) and (5) that

$$C_{11}C_{11}^* = D_{11}. \tag{6}$$

Denoting by c_i, $(i = 1, 2, \cdots, r)$, the row vectors of C_{11}, and by d_i the element in the ith place of the diagonal matrix D_{11}, we

† Turnbull and Aitken, op. cit., p. 85.

deduce the equalities $c_i c_j^* = 0$, $(i \neq j)$, $c_i c_i^* = d_i$, $(i, j = 1, 2, \cdots, r)$. Hence d_i is a positive real number and the vector $c_i/d_i^{1/2}$ is a normalized vector. Consequently the matrix

$$V_{11} = \begin{pmatrix} d_1^{-1/2} & 0 & \cdots & 0 \\ 0 & d_2^{-1/2} & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & d_r^{-1/2} \end{pmatrix} C_{11} = (D_{11})^{-1/2} C_{11}$$

is unitary and

$$C_{11} = Q_{11} V_{11}, \tag{7}$$

where V_{11} is a unitary matrix of order r and $Q_{11} = (D_{11})^{1/2}$ is a positive definite hermitian matrix of order r. Using the value of C given by (5), we have, from (7),

$$\begin{aligned} C &= \begin{pmatrix} Q_{11} V_{11} & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{pmatrix} \\ &= \begin{pmatrix} Q_{11} & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{pmatrix} \begin{pmatrix} V_{11} & 0_{r,n-r} \\ 0_{n-r,r} & E_{n-r} \end{pmatrix}. \end{aligned} \tag{8}$$

Now if $m \leqq n$, this last result may be written in the form

$$C = (Q \quad 0_{m,n-m}) V, \tag{9}$$

where

$$Q = \begin{pmatrix} Q_{11} & 0_{r,m-r} \\ 0_{m-r} & 0_{m-r,m-r} \end{pmatrix}, \qquad V = \begin{pmatrix} V_{11} & 0_{r,n-r} \\ 0_{n-r,r} & E_{n-r} \end{pmatrix},$$

Q being a positive hermitian matrix of order m and rank r, while V is a unitary matrix of order n. Moreover, if

$$C = (Q_1 \quad 0_{m,n-m}) V_1 \tag{10}$$

is another such representation of C, where Q_1 is a positive hermitian matrix of order m and rank r and V_1 is a unitary matrix of order n, we see that $CC^* = Q_1 Q_1^* = Q_1^2 = D$, so that $Q_1 = D^{1/2} = Q$. Accordingly we may write (10) in the form

$$C = \begin{pmatrix} Q_{11} & 0_{r,n-r} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{pmatrix} \begin{pmatrix} W_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix},$$

where W_{11} is an r-rowed square matrix, V_{22} an $(n-r)$-rowed square matrix, V_{12} an r by $n-r$ and V_{21} an $n-r$ by n matrix. Hence

$$C = \begin{pmatrix} Q_{11}W_{11} & Q_{11}V_{12} \\ 0_{m-r,r} & 0_{m-r,n-r} \end{pmatrix},$$

and comparison with (8) shows that $W_{11}=V_{11}$ and $V_{12}=0_{r,n-r}$. Since $V_1V_1^*=E_n$, it follows easily that $V_{21}=0_{n-r,r}$ and that $V_{22}V_{22}^*=E_{n-r}$. Hence the matrix V_1 in (10) is of the form VW, where

$$W = \begin{pmatrix} E_r & 0_{r,n-r} \\ 0_{n-r,r} & V_{22} \end{pmatrix}$$

and V_{22} is an arbitrary unitary matrix of order $n-r$. We have therefore proved the following lemma.

LEMMA. *The matrix C can be represented in the form $C=(Q\ \ 0_{n-m})\ V_1$, where Q is a positive hermitian matrix of order m and rank r and V_1 is a unitary matrix of order n. The matrix Q is unique while the matrix V_1 is one of a set $[V_1]=[VW]$, where V is a fixed unitary matrix and W ranges over a group G of unitary matrices of order n simply isomorphic with the group of all unitary matrices of order $n-r$.*

Since $A=X^*CY^*$, by (9),

$$A = X^*(Q \quad 0_{m,n-m})VY^* = X^*(Q \quad 0_{m,n-m})X_1X_1^*VY^*,$$

where

$$X_1 = \begin{pmatrix} X & 0_{m,n-m} \\ 0_{n-m,m} & E_{n-m} \end{pmatrix}.$$

Hence

$$A = (X^*QX \quad 0_{m,n-m})X_1^*VY^* = (P \quad 0_{m,n-m})U,$$

where $P=X^*QX$ is a positive hermitian matrix of order m and rank r, while $U=X_1^*VY^*$ is a unitary matrix of order n. If $A=(P_1\ \ 0_{m,n-m})U_1$ is another such representation of A, it follows easily from the previous lemma that $XP_1X^*=Q$, so that $P_1=P$, and that $U_1=UZ$, where $Z=YWY^*$. Accordingly we have proved the following theorem.

THEOREM. *If A is a matrix of m rows and n columns of rank r and $m \leqq n$, A can be represented in the form*

$$A = (P_1 \quad 0_{m,n-m})U_1, \tag{11}$$

where P_1 is a positive hermitian matrix of order m and rank r and U_1 is a unitary matrix of order n. The matrix P_1 is unique while the matrix U_1 is one of a set $[U_1] = [UZ_1]$, where U is a fixed unitary matrix and Z_1 one of a group G_1 of matrices, simply isomorphic with the group of all unitary matrices of order $n-r$.

COROLLARY. *Under the above hypotheses the totality of unitary matrices Z_1 for which $AZ_1 = A$ forms a group simply isomorphic with the group of all unitary matrices of order $n-r$. This group is the group G_1.*

For if $AZ_1 = A$, $(P_1 \quad 0_{m,n-m})UZ_1 = (P_1 \quad 0_{m,n-m})U$, and Z^1 must lie in G_1. Similarly if Z_1 lies in G_1, $AZ_1 = A$.

If $m \geqq n$, A^* is a matrix in which the number of its rows is at most equal to the number of its columns. Accordingly, under this hypothesis our theorem is true if A is replaced by A^*. Hence, if $m \geqq n$, A can be represented in the form

$$A = U_2 \begin{pmatrix} P_2 \\ 0_{m-n,n} \end{pmatrix}, \tag{12}$$

where P_2 is a uniquely determined positive hermitian matrix of order n and rank r and U_2 is one of a set $[U_2] = [Z_2U]$, where Z_2 ranges over a group G_2 simply isomorphic with the group of all unitary matrices of order $m-r$.

When $m = n$, that is, when the matrix A is square, some further results follow. In this case equations (11) and (12) become

$$A = P_1U_1, \tag{13}$$

and

$$A = U_2P_2, \tag{14}$$

respectively, where P_1 and P_2 are positive hermitian matrices of rank r and U_1 and U_2 are unitary matrices. *The two groups G_1 and G_2 are simply isomorphic; the two sets $[U_1]$ and $[U_2]$ coincide; if U_1 belongs to the set $[U_1]$, $P_2 = U_1^*P_1U_1$.* The first

statement is obviously true; if U_1 lies in $[U_1]$, $A = U_1U_1^*P_1U_1$ and, since $U_1^*P_1U_1$ is a positive hermitian matrix of rank r, $U_1^*P_1U_1 = P_2$ and U_1 lies in $[U_2]$. Similarly any member U_2 of $[U_2]$ lies in $[U_1]$. Further *the matrix P_2 is invariant under unitary transformation by any matrix of the group G_1, and P_1 under transformation by any matrix of the group G_2.* For if Z_1 lies in G_1, $AZ_1 = A$ so that $A = U_2Z_1Z_1^*P_2Z_1$, and accordingly, $Z_1^*P_2Z_1 = P_2$.

THE JOHNS HOPKINS UNIVERSITY

ON A THEOREM OF FÉRAUD

BY D. C. LEWIS, JR.*

The Birkhoff-Pfaffian equations of dynamics are written in variational form as follows:

$$\delta \int \left[\sum_{i=1}^{2m} X_i \left(\frac{dx_i}{dt} \right) + Q \right] dt = 0,$$

where Q and the X's are functions of $x_1, \cdots, x_{2m}$ and, in general, depend also periodically upon t, and where the skew-symmetric determinant $|a_{ij}|$, $(a_{ij} = \partial X_i/\partial x_j - \partial X_j/\partial x_i)$, does not vanish in the regions considered. We restrict attention to the neighborhood of a generalized equilibrium point, that is, a point where all the $\partial Q/\partial x_i - \partial X_i/\partial t$ vanish identically in t. We take this point at the origin, $x_i = 0$, $(i = 1, 2, \cdots, 2m)$.

The problem of reducing the Pfaffian system to a Hamiltonian system can be reduced to that of finding a non-singular transformation, $x_i = x_i(y_1, \cdots, y_{2m})$, leaving the origin invariant (and depending in general periodically upon t) which reduces the linear differential form $\sum_{i=1}^{2m} X_i dx_i$ to the form $\sum_{i=1}^{m} y_{2i} dy_{2i-1} + dw$, where dw is an exact differential in $y_1, \cdots, y_{2m}$, the coefficients of which are independent of t. This same problem also will play an important role in a future paper of mine on "conservative" transformations in $2m$-dimensional spaces.

The problem has been considered by Féraud,† who obtained a

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* National Research Fellow.

† *Extension au cas d'un nombre quelconque de degrés de liberté d'une propriété relative aux systèmes Pfaffiens*, Comptes Rendus, vol. 190 (1930), pp. 358–360.

solution based on the fundamental existence theorem of Riquier for systems of partial differential equations. It appears, however, highly desirable to give a more elementary proof based on existence theorems for *ordinary* differential equations. The result is thus not limited to the analytic case, as it was by the use of Riquier's theorem in Féraud's proof.

Our proof follows a well known line of thought expounded by Goursat.* There are several reasons, however, why Goursat's writings on the reduction of linear differential forms are not immediately applicable. For one thing, the transformations used by him do not necessarily leave the origin invariant. Also there are important cases where his reasoning might fail.† The present treatment does not presuppose a knowledge of Goursat's work.

We split up our proof into a number of easy lemmas.

LEMMA 1. *Consider a linear differential form* $\sum_{i=1}^{n} X_i dx_i$ *which under a non-singular transformation of coordinates,* $x_i = g_i(y_1, \cdots, y_n)$, *appears also in the form* $\sum_{i=1}^{n} Y_i dy_i$. *Then we have the following elementary relations*:

$$Y_i = \sum_{j=1}^{n} X_j \left(\frac{\partial g_j}{\partial y_i}\right), \tag{1}$$

$$\frac{\partial Y_i}{\partial y_r} - \frac{\partial Y_r}{\partial y_i} = \sum_{l,j=1}^{n} \left(\frac{\partial X_j}{\partial x_l} - \frac{\partial X_l}{\partial x_j}\right) \frac{\partial g_l}{\partial y_r} \frac{\partial g_j}{\partial y_i}. \tag{2}$$

The proof of (1) is trivial, while (2) is deduced at once from (1) by differentiation with respect to y_r, interchange of indices, and subtraction. In the sequel, we shall uniformly set $a_{ij} = \partial X_i/\partial x_j - \partial X_j/\partial x_i$, and $b_{ij} = \partial Y_i/\partial y_j - \partial Y_j/\partial y_i$; so that (2) may also be written in the form

$$b_{ij} = \sum_{k,l=1}^{n} a_{kl} \frac{\partial g_k}{\partial y_i} \frac{\partial g_l}{\partial y_j}. \tag{3}$$

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* *Leçons sur le Problème de Pfaff*, 1922, especially Chapter 1.

† For example, if the first method of Chapter I is used, care must be taken, in the series of necessary reductions, that at each step the λ_i (pages 11 and 12) do not all vanish at the origin. The present paper does, in effect, just this. If the more advanced method of Chapter 4 is used, it would be necessary to compute the derived forms. Also the most available results of Chapter 4 are stated only for the analytic case.

LEMMA 2. *Consider the system of ordinary differential equations*

$$\frac{dx_i}{dy_n} = f_i(x_1, \cdots, x_n), \qquad (i = 1, 2, \cdots, n),$$

where the f_i are defined and of class C^r in the neighborhood of the origin. Let $g_i(y_1, \cdots, y_{n-1}, y_n)$ be the solution which satisfies the initial conditions $x_1 = y_1, \cdots, x_{n-1} = y_{n-1}, x_n = 0$ for $y_n = 0$. Then the g's are of class C^r in the neighborhood of the origin, and, if the equations $x_i = g_i(y_1, \cdots, y_{n-1}, y_n)$ are thought of as defining a transformation T of the neighborhood of the origin into itself, then T is non-singular if $f_n(0, \cdots, 0) \neq 0$.

The fact that the g's are of class C^r is an immediate consequence of a well known theorem.* The rest of the lemma follows from the obvious relations

$$(4) \qquad \begin{aligned} &\left.\frac{\partial g_i}{\partial y_j}\right|_0 = \delta_{ij}, && (i = 1, 2, \cdots, n), \\ &\left.\frac{\partial g_i}{\partial y_j}\right|_0 = f_i(0, \cdots, 0), && (j = 1, 2, \cdots, n-1). \end{aligned}$$

For the jacobian of T, evaluated at the origin, turns out to be exactly $f_n(0, \cdots, 0)$.

LEMMA 3. *Consider a Pfaffian $\omega = \sum_{i=1}^{2k} X_i dx_i$ satisfying the following conditions:*

(A) *The X's are of class C^μ in the neighborhood of the origin.*

(B) *None of the X's vanish at the origin.*

(C) *The skew-symmetric determinant $|a_{ij}|$ does not vanish at the origin.*

Then there exists a non-singular transformation $x_i = g_i(y_1, \cdots, y_{2k})$ of the neighborhood of the origin into itself such that ω appears expressed in terms of the y's in the form $e^{y_{2k}}[\sum_{i=1}^{2k-1} A_i dy_i]$, where the A's are functions of $y_1, \cdots, y_{2k-1}$ satisfying the following conditions:

(α) *The A_i are of class $C^{\mu-2}$.*

(β) *None of the A_i vanish at the origin.*

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* See G. A. Bliss, *The solutions of differential equations of the first order as functions of their initial values*, Annals of Mathematics, vol. 6 (1904–1905), pp. 49–68, especially p. 67.

(γ) *The skew-symmetric matrix* (c_{ij}), *where* $c_{ij}=\partial A_i/\partial y_j - \partial A_j/\partial y_i$, *is such that there exists a system of functions* $\xi_i(y_1, \cdots, y_{2k-1})$ *which satisfy the linear identities* $\sum_{j=1}^{2k-1} c_{ij}\xi_j \equiv 0$ *and* $\sum_{j=1}^{2k-1} A_j\xi_j \equiv 1$.

PROOF. Consider the equations

$$\sum_{j=1}^{2k} a_{ij}\left(\frac{dx_j}{dy_{2k}}\right) = X_i, \qquad (i = 1, 2, \cdots, 2k). \tag{5}$$

Since $|a_{ij}| \neq 0$, we may solve for the derivatives, writing the result in the form $dx_i/dy_{2k}=f_i(x_1, \cdots, x_{2k})$, where the f's are of class $C^{\mu-1}$. On account of (5) and (B) it is seen that not all the f's vanish at the origin. It is no loss of generality to assume $f_{2k}(0, \cdots, 0)\neq 0$, since a preliminary linear transformation can always be applied, the effect of which is merely to interchange indices in the desired manner. We now apply Lemma 2 to the present system, taking $n=2k$ and $\nu=\mu-1$. We proceed to show that the functions g_i, as there defined, satisfy the conditions of the present lemma.

In equations (5) replace dx_j/dy_{2k} by $\partial g_j/\partial y_{2k}$. Multiply each equation by $\partial g_i/\partial y_{2k}$ and sum with respect to i. Since $a_{ij}+a_{ji}=0$, the result is $\sum X_i(\partial g_i/\partial y_{2k})=0$. Hence from (1) we see that

$$Y_{2k} \equiv 0. \tag{6}$$

We also have, of course, $\partial Y_{2k}/\partial y_i \equiv 0$, so that (2) yields

$$\frac{\partial Y_i}{\partial y_{2k}} = \sum_{l,j=1} a_{jl}\frac{\partial g_l}{\partial y_{2k}}\frac{\partial g_j}{\partial y_i},$$

and, since the g's satisfy (5), the right member is equal to $\sum X_j(\partial g_j/\partial y_i)$, which by (1) is precisely Y_i. Hence we have the result that $\partial Y_i/\partial y_{2k} \equiv Y_i$, or

$$Y_i = e^{y_{2k}}A_i, \tag{7}$$

where A_i is a suitably chosen function of $y_1, \cdots, y_{2k-1}$, but independent of y_{2k}. From (6) and (7) we have

$$\omega = \sum_{i=1}^{2k} Y_i dy_i = e^{y_{2k}}\Big[\sum_{i=1}^{2k-1} A_i dy_i\Big].$$

Now the g's are of class $C^{\mu-1}$ by Lemma 2. Hence the Y's and, therefore, the A's, are of class $C^{\mu-2}$ by (1). By (7), (1), and (4),

we have $A_i = Y_i = \sum_{j+1}^{2k} X_j \delta_{ji} = X_i$, $(i = 1, 2, \cdots, 2k-1)$, at the origin. Hence (β) follows at once from (B).

It remains to show that the A's as defined above satisfy (C). It follows from (3) that

$$|b_{ij}| = |a_{ij}| \cdot \left(\frac{\partial(g_1, \cdots, g_{2k})}{\partial(y_1, \cdots, y_{2k})}\right)^2 \neq 0.$$

Hence we have a unique set of functions $\xi_1, \cdots, \xi_{2k}$ satisfying the equations

$$\sum_{j=1}^{2k-1} b_{ij}\xi_j + b_{i,2k}\xi_{2k} = 0, \qquad (i = 1, 2, \cdots, 2k-1),$$

$$\sum_{j=1}^{2k-1} - b_{j,2k}\xi_j + 0 = - e^{v_{2k}}, \qquad (\text{since } b_{2k,j} = - b_{j,2k}).$$

Now

$$b_{j,2k} = \frac{\partial Y_j}{\partial y_{2k}} - \frac{\partial Y_{2k}}{\partial y_j} = Y_j = e^{v_{2k}} A_j, \qquad (j = 1, 2, \cdots, 2k-1),$$

and

$$b_{ij} = e^{v_{2k}}\left(\frac{\partial A_i}{\partial y_j} - \frac{\partial A_j}{\partial y_i}\right) = e^{v_{2k}} c_{ij}, \quad (i, j = 1, 2, \cdots, 2k-1),$$

on account of (6) and (7). After dividing through by $e^{v_{2k}}$, our system of $2k$ linear equations may, therefore, be written in the form,

$$\sum_{j=1}^{2k-1} c_{ij}\xi_j + A_i\xi_{2k} = 0, \qquad \sum_{j=1}^{2k-1} A_j\xi_j = 1.$$

The coefficients now are independent of y_{2k}. Furthermore $\xi_{2k} = 0$, as is easy to see from Cramer's rule, since the odd order determinant $|c_{ij}|$ is skew-symmetric and therefore identically zero. Hence condition (γ) is fulfilled.

LEMMA 4. *Consider a Pfaffian* $\omega = \sum_{i=1}^{2k-1} X_i dx_i$ *satisfying the following conditions*:

(A) *The X's are of class* C^a.

(B) *None of the X's vanish at the origin.*

(C) *The matrix* (a_{ij}) *is such that there exists a system of functions,* $\xi_i(x_1, \cdots, x_{2k-1})$, *which satisfy the linear equations*

$$\sum_{j=1}^{2k-1} a_{ij}\xi_j = 0, \qquad \sum_{j=1}^{2k-1} X_j\xi_j = 1.$$

Then there exists a non-singular transformation $x_i = g_i$ $(y_1, \cdots, y_{2k-1})$ *of the neighborhood of the origin into itself such that* ω *appears in terms of the y's in the form* $\sum_{i=1}^{2k-2} Y_i dy_i + dy_{2k-1}$, *where the* Y_i *are functions of* $y_1, \cdots, y_{2k-2}$ *(independent of* y_{2k-1}*) satisfying the following conditions:*

(α) *The Y's are of class* $C^{\mu-2}$.

(β) *None of the Y's vanish at the origin.*

PROOF. Consider the equations

$$\frac{dx_i}{dy_{2k-1}} = \xi_i(x_1, \cdots, x_{2k-1}), \qquad (i = 1, 2, \cdots, 2k-1). \tag{8}$$

It is clear that at least one of the ξ's does not vanish at the origin on account of (C). Without loss of generality we assume that $\xi_{2k-1}(0, \cdots, 0) \neq 0$. It is also clear that the ξ's may be taken to be of class $C^{\mu-1}$. We can apply Lemma 2 to (8), taking $f_i = \xi_i$, $n = 2k-1$, $\nu = \mu - 1$. We show that the g_i, as there defined, satisfy the conditions of the present lemma.

For, on account of (8) and (C), we have

$$\sum_{j=1}^{2k-1} X_j\left(\frac{\partial g_j}{\partial y_{2k-1}}\right) = 1, \qquad \sum_{j=1}^{2k-1} a_{ij}\left(\frac{\partial g_j}{\partial y_{2k-1}}\right) = 0.$$

Hence, referring back to (1), we have $Y_{2k-1} \equiv 1$, while (2) yields the result that

$$\frac{\partial Y_i}{\partial y_{2k-1}} = \frac{\partial Y_i}{\partial y_{2k-1}} - \frac{\partial Y_{2k-1}}{\partial y_i} = \sum_{j,l=1}^{2k-1} a_{jl}\frac{\partial g_j}{\partial y_i}\frac{\partial g_l}{\partial y_{2k-1}}$$

$$= \sum_{j=1}^{2k-1}\left[\sum_{l=1}^{2k-1} a_{jl}\left(\frac{\partial g_l}{\partial y_{2k-1}}\right)\right]\left(\frac{\partial g_j}{\partial y_i}\right) = 0.$$

This shows that the Y's are all independent of y_{2k-1}. It remains to prove that conditions (α) and (β) are satisfied.

Condition (α) follows from (1) and the fact that, by Lemma

2, the g's are of class $C^{\mu-1}$. Condition (β) follows from the fact that, by (1) and (4), $X_i|_0 = Y_i|_0$, $(i=1, 2, \cdots, 2k-2)$.

We are now ready to prove our main result.

THEOREM. *Consider a Pfaffian* $\omega=\sum_{i=1}^{2m} X_i dx_i$ *satisfying the following conditions*:

(I) *The* X*'s are of class* C^{4m+p}, $(p \geqq 0)$.

(II) *None of the* X*'s vanish at the origin.*

(III) *The determinant* $|a_{ij}|$ *does not vanish at the origin.*

Then there exists a non-singular transformation of coordinates, valid in the vicinity of the origin, which it leaves fixed, such that ω *appears in the form* $\sum_{i=1}^{m} y_{2i}dy_{2i-1}+d(\sum_{i=1}^{m}y_{2i-1})$. *This transformation is of class* C^{p+1}.

PROOF. Applying Lemma 3, with $k=m$ and $\mu=4m+p$, we write ω in the form $e^{x_{2m}}(\sum_{i=1}^{2m-1}X_i dx_i)$, where the new X's are independent of x_{2m} and in fact satisfy all the hypotheses of Lemma 4, with $k=m$ and $\mu=4m-2+p$. Hence, applying Lemma 4, we can further simplify ω, which now appears in the form

$$e^{x_{2m}}\left(\sum_{i=1}^{2m-2} X_i dx_i\right) + e^{x_{2m}}dx_{2m-1}.$$

The new X's, by Lemma 4, satisfy all the hypotheses of Lemma 3 (with $k=m-1$, $\mu=4(m-1)+p$), except that we must verify that the new determinant $|a_{ij}|$ does not vanish. This, however, can easily be done with the help of (3) on account of the fact that the transformation in question is known to be non-singular. Hence we can successively apply Lemmas 3 and 4 again and as a result we have ω in the form

$$e^{x_{2m}}e^{x_{2m-2}}\left(\sum_{i=1}^{2m-4} X_i dx_i\right) + e^{x_{2m}}e^{x_{2m-2}}dx_{2m-3} + e^{x_{2m}}dx_{2m-1}.$$

The new X's are independent of $x_{2m-3}, \cdots, x_{2m}$ and are of class $C^{4(m-2)+p}$. This process may obviously be continued until at the end of the mth application of Lemmas 3 and 4 we get ω in the form

$$e^{x_2}e^{x_4}\cdots e^{x_{2m}}dx_1 + \cdots + e^{x_{2m}}e^{x_{2m-2}}dx_{2m-3} + e^{x_{2m}}dx_{2m-1}.$$

The transformation at the last application of Lemma 4 is of

class C^{p+1}. The transformations at the previous steps are of higher class. Hence the compound transformation is of class C^{p+1}. Finally we make the additional transformation:

$$y_1 = x_1, \qquad y_2 = e^{x_2}e^{x_4}\cdots e^{x_{2m}} - 1,$$
$$y_3 = x_3, \qquad y_4 = e^{x_4}e^{x_6}\cdots e^{x_{2m}} - 1,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$y_{2m-1} = x_{2m-1}, \qquad y_{2m} = e^{x_{2m}} - 1.$$

This transformation, which obviously is analytic with jacobian equal to 1 at the origin, leaves the origin invariant and transforms ω into the form $(y_2+1)dy_1+\cdots+(y_{2m-2}+1)dy_{2m-3}+(y_{2m}+1)dy_{2m-1}$. This completes the proof.

It is scarcely necessary to add that, if (I) be replaced by the hypothesis that the X's be analytic, the transformation T furnishing the desired reduction of ω will also be analytic. To see this it is only necessary to make slight obvious changes in the previous work, beginning with Lemma 2, where we take the f's and consequently the g's as analytic.

In the application to the reduction of the Pfaffian equations of dynamics to Hamiltonian form, the condition (II) is of no disadvantage. For the Pfaffian differential equations are unaltered if to each X_i a constant is added. Assuming then that (II) holds (for each value of t) the equations may be written in variational form as follows:

$$\delta\int\left[\sum_{i=1}^{m} y_{2i}\left(\frac{dy_{2i-1}}{dt}\right) + H\right]dt = 0,$$

where now, since the X's (and consequently T) depend in general periodically upon t, H is not necessarily equal to Q, but rather, of course,

$$H = Q + \sum_{i=1}^{2m} X_i\left(\frac{dx_i}{dt}\right).$$

PRINCETON UNIVERSITY

ON THE rTH DERIVED CONJUGATE FUNCTION*

BY A. F. MOURSUND

1. *Introduction.* We assume throughout this note that the function $f(x)$ is Lebesgue integrable on $(-\pi, \pi)$ and of period 2π; then the series

$$(1) \qquad \sum_{n=1}^{\infty} (-b_n \cos nx + a_n \sin nx),$$

where a_n and b_n are the Fourier coefficients, is known as the conjugate Fourier series.

It is customary to associate with the series (1) as sum or "conventionalized" sum the conjugate function of $f(x)$ which is either the limit

$$(2) \qquad \tilde{f}_1(x) \equiv \frac{-1}{2\pi} \lim_{\epsilon \to 0} \int_{\epsilon}^{\pi} \{f(x+s) - f(x-s)\} \operatorname{ctn} \frac{s}{2} ds,$$

or the equivalent limit

$$(3) \qquad \tilde{f}_2(x) \equiv \frac{-1}{\pi} \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{f(x+s) - f(x-s)}{s} ds;$$

and to associate with the first derived series of the series (1) the first derived conjugate function which is either the limit

$$(4) \qquad \tilde{f}_1'(x) \equiv \frac{-1}{4\pi} \lim_{\epsilon \to 0} \int_{\epsilon}^{\pi} \{f(x+s) + f(x-s) - 2f(x)\} \csc^2 \frac{s}{2} ds,$$

or the equivalent limit

$$(5) \qquad \tilde{f}_2'(x) \equiv \frac{-1}{\pi} \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{f(x+s) + f(x-s) - 2f(x)}{s^2} ds.\dagger$$

* Presented to the Society, December 1, 1934.

† For a proof of the equivalence of (2) and (3) see G. H. Hardy and J. E. Littlewood, *The allied series of a Fourier series*, Proceedings of the London Mathematical Society, vol. 24 (1925), pp. 211–246 (p. 221). A proof of the simultaneous existence of (4) and (5) is given by A. H. Smith, *On the summability of derived series of the Fourier-Lebesgue type*, Quarterly Journal of Mathematics, (Oxford Series), vol. 4 (1933), pp. 93–106 (p. 106); that they are equivalent follows from Theorem 1 of this note.

The limits (2) and (3) and, provided $f(x)$ is of bounded variation, the limits (4) and (5) exist almost everywhere.*

In a recent paper,† generalizing (2) and (4), we defined the rth derived conjugate function as follows.

DEFINITION 1.

$$\tilde{f}_1^{(r)}(x) \equiv \frac{(-1)^{r+1}}{2\pi} \lim_{\epsilon \to 0} \int_{\epsilon}^{\pi} A_r(s) \frac{d^r \operatorname{ctn}(s/2)}{ds^r} ds - C_r,$$

where $f^{(i)}(x)$ designates the ith generalized derivative of $f(x)$,

$$A_r(s) \equiv f(x+s) + (-1)^{r+1} f(x-s) - 2 \sum_{i=0}^{[(r-1)/2]} \frac{s^{r-1-2i}}{(r-1-2i)!} f^{(r-1-2i)}(x), \tag{6}$$

and

$$C_r \equiv \sum_{i=0}^{[r/2]-1} f^{(r-1-2i)}(x) \sum_{j=i}^{[r/2]-1} \frac{\pi^{2j-2i}}{(2j+1-2i)!} \frac{d^{2j+1} \operatorname{ctn}(s/2)}{ds^{2j+1}} \Bigg]_{s=\pi}. \tag{7}$$

We used the limit $\tilde{f}_1^{(r)}(x)$, which (i) exists wherever $f^{(r+1)}(x)$ exists and (ii) exists almost everywhere when $d^{r-1}f(x)/dx^{r-1}$ is of bounded variation, as the $N_{z_{r+1}}$ sum of the rth derived series of the series (1).‡

For use as a sum function in theorems concerning the summability of the rth derived series of the series (1) by certain standard methods, among which is the first form of the Bosanquet-Linfoot (α, β) method,§ it is desirable to define an rth derived conjugate function of the type (3), (5). In this note

* A. Plessner, *Zur Theorie der konjugierten trigonometrischen Reihen*, Mitteilungen des Mathematischen Seminars der Universität Giessen, Heft 10 (1923).

† A. F. Moursund, *On summation of derived series of the conjugate Fourier series*, Annals of Mathematics, (2), vol. 36 (1935), p. 184. Throughout this note $[r]$ is the greatest integer $\leqq r$. Hence $C_0, C_1 = 0$. For the definition and properties of generalized derivatives see Ch. de la Vallée-Poussin, *Approximation des fonctions par des polynomes*, Académie Royale de Belgique, Bulletins, Classe des Sciences, (1908), pp. 195–254 (p. 214).

‡ See loc. cit. The N_{z_p} method includes as a special case the second form of the summation method defined by L. S. Bosanquet and E. H. Linfoot in *Generalized means and the summability of Fourier series*, Quarterly Journal of Mathematics, (Oxford Series), vol. 2 (1931), pp. 207–229.

§ Loc. cit., p. 208.

we define such an rth derived conjugate function, prove that the function is equivalent to $\tilde{f}_1^{(r)}(x)$, and use it as the Bosanquet-Linfoot sum of the rth derived series of the series (1).*

2. *Definition of $\tilde{f}_2^{(r)}(x)$. Theorems.* The function $A_r(s)$ is defined in the introduction only for $0 \leqq s \leqq \pi$. We now set for s on $(-\infty, \infty)$

$$A_r(s) \equiv f(x+s) + (-1)^{r+1} f(x-s) - 2 \sum_{i=0}^{[(r-1)/2]} \frac{\{p(s)\}^{r-1-2i}}{(r-1-2i)!} f^{(r-1-2i)}(x), \tag{8}$$

where $p(s) = s - 2k\pi$ for $(2k-1)\pi < s \leqq (2k+1)\pi$; and we define $\tilde{f}_2^{(r)}(x)$ as follows.

DEFINITION 2.

$$\tilde{f}_2^{(r)}(x) \equiv \frac{-r!}{\pi} \lim_{\epsilon \to 0} \int_\epsilon^\infty \frac{A_r(s)}{s^{r+1}}\, ds - C_r.$$

The following theorems give our principal results.

THEOREM 1. *Whenever either of the limits $\tilde{f}_1^{(r)}(x), \tilde{f}_2^{(r)}(x)$ exists, the other exists and the two are equal.*

THEOREM 2. *The rth derived series of the series (1) is summable to $\tilde{f}_2^{(r)}(x)$ by the Bosanquet-Linfoot (α, β) method with $\alpha = r$, $\beta > 1$ or $\alpha > r$ wherever $\tilde{f}_2^{(r)}(x)$ exists and $\int_0^s |A_r(t)|\, dt = o(s^{r+1})$, as $s \to 0$.*†

3. *Proof of Theorem* 1. We use the following lemma in the proofs of both Theorems 1 and 2.

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* An rth derived conjugate function of the type (3), (5) is defined by A. H. Smith in a recent paper, *On the summability of derived conjugate series of the Fourier-Lebesgue type*, this Bulletin, vol. 40 (1934), pp. 406–412; however, it is impossible to give good conditions for the existence of his function. A simple example will serve to show this. Let $f(x) = \sin x$; then the second derived conjugate series is $\cos x$ and converges to 1 when $x = 0$, while for $x = 0$, Smith's $g^{(2)}(x) = \lim_{\epsilon \to 0} -(4/\pi) \int_\epsilon^\infty (\sin t/t^2)\, dt$ and does not exist. All further references to Smith are to this paper.

† We give this theorem primarily to illustrate the use of the function $\tilde{f}_2^{(r)}(x)$. The condition $\int_0^s |A_r(t)|\, dt = o(s^{r+1})$ is equivalent to the condition $\int_0^s (|A_r(t)|/t^r)\, dt = o(s)$; see Smith, loc. cit., pp. 411–412.

LEMMA 1. *For s on $(-\pi, \pi)$ the series*

$$\sum_{-\infty}^{\infty}{}' \frac{(-1)^r r!}{(s+2k\pi)^{r+1}} \equiv \lim_{k\to\infty}\left(\sum_{-k}^{-1} + \sum_{1}^{k}\right)$$

converges uniformly to the function

$$\frac{1}{2}\frac{d^r \operatorname{ctn}(s/2)}{ds^r} + \frac{(-1)^{r+1} r!}{s^{r+1}}.$$

PROOF. The lemma follows upon differentiating termwise r times the series $\sum_{-\infty}^{\infty} 1/(s+2k\pi)$ which converges uniformly on $(-\pi, \pi)$ to the function $(1/2)\operatorname{ctn}(s/2) - 1/s$.*

PROOF OF THEOREM 1.† Since $A_r(s)$ is of period 2π and the functions

$$\frac{A_r(s)}{s^{r+1}}, \qquad A_r(s)\left\{\frac{1}{2}\frac{d^r \operatorname{ctn}(s/2)}{ds^r} + \frac{(-1)^{r+1} r!}{s^{r+1}}\right\},$$

are even functions of s, we have, using Lemma 1,

$$\begin{aligned}
\int_{\epsilon}^{\infty} \frac{A_r(s)}{s^{r+1}}\,ds &= \int_{\epsilon}^{\pi} + \int_{\pi}^{\infty} \\
&= \int_{\epsilon}^{\pi} \frac{A_r(s)}{s^{r+1}}\,ds + \frac{1}{2}\sum_{-\infty}^{\infty}{}' \int_{(2k-1)\pi}^{(2k+1)\pi} \frac{A_r(s)}{s^{r+1}}\,ds \\
&= \int_{\epsilon}^{\pi} \frac{A_r(s)}{s^{r+1}}\,ds + \frac{1}{2}\sum_{-\infty}^{\infty}{}' \int_{-\pi}^{\pi} \frac{A_r(s)}{(s+2k\pi)^{r+1}}\,ds \\
&= \int_{\epsilon}^{\pi} \frac{A_r(s)}{s^{r+1}}\,ds \\
&\qquad + \frac{(-1)^r}{2r!}\int_{-\pi}^{\pi} A_r(s)\left\{\frac{1}{2}\frac{d^r \operatorname{ctn}(s/2)}{ds^r} + \frac{(-1)^{r+1} r!}{s^{r+1}}\right\} ds \\
&= \int_{\epsilon}^{\pi} \frac{A_r(s)}{s^{r+1}}\,ds \\
&\qquad + \frac{(-1)^r}{r!}\int_{0}^{\pi} A_r(s)\left\{\frac{1}{2}\frac{d^r \operatorname{ctn}(s/2)}{ds^r} + \frac{(-1)^{r+1} r!}{s^{r+1}}\right\} ds
\end{aligned}$$

* T. J. I'a. Bromwich, *Theory of Infinite Series*, 1926 edition, pp. 216–218.

† This proof was suggested to us by the proof of the equivalence of $\bar{f}_1(x)$ and $\bar{f}_2(x)$ given by Hardy and Littlewood, loc. cit., p. 221.

$$= \frac{(-1)^r}{2r!}\int_\epsilon^\pi A_r(s)\frac{d^r \operatorname{ctn}(s/2)}{ds^r}\,ds$$

$$+ \frac{(-1)^r}{r!}\int_0^\epsilon A_r(s)\left\{\frac{1}{2}\frac{d^r \operatorname{ctn}(s/2)}{ds^r} + \frac{(-1)^{r+1}r!}{s^{r+1}}\right\}ds$$

$$= \frac{(-1)^r}{2r!}\int_\epsilon^\pi A_r(s)\frac{d^r \operatorname{ctn}(s/2)}{ds^r}\,ds + o(1) \text{ as } \epsilon \to 0.$$

The theorem follows when we multiply by the constant factor $-r!/\pi$.

4. *Proof of Theorem* 2. We base the proof of Theorem 2 upon the proof of an analogous theorem given by A. H. Smith, and use as far as possible his notation.* The use of $\bar{f}_2^{(r)}(x)$ instead of Smith's $g^{(r)}(x)$ as sum for the rth derived conjugate Fourier series complicates the proof of our theorem.†

In addition to Smith's lemmas we need the following fundamental lemma.

LEMMA 2. *For* $\beta > 0$,

$$2(-1)^r \sum_{i=0}^{[(r-1)/2]} \frac{f^{(r-1-2i)}(x)}{(r-1-2i)!}\int_0^\infty \{p(t)\}^{r-1-2i}\,\bar{\lambda}_{r+1,\beta}^{(r)}(n, t)\,dt$$

$$= C_r + o(1) \text{ as } n \to \infty.$$

PROOF. Using Smith's Lemmas 1 and 2 and our Lemma 1, we have‡

$$\int_0^\infty \{p(t)\}^{r-1-2i}\,\bar{\lambda}_{r+1,\beta}^{(r)}(n, t)\,dt = \frac{1}{2}\int_{-\infty}^\infty \{p(t)\}^{r-1-2i}\,\bar{\lambda}_{r+1,\beta}^{(r)}(n, t)\,dt$$

$$= \frac{1}{2}\sum_{-\infty}^{\infty}\int_{(2k-1)\pi}^{(2k+1)\pi} (t - 2k\pi)^{r-1-2i}\,\bar{\lambda}_{r+1,\beta}^{(r)}(n, t)\,dt$$

$$= \frac{1}{2}\sum_{-\infty}^{\infty}\int_{-\pi}^{\pi} t^{r-1-2i}\,\bar{\lambda}_{r+1,\beta}^{(r)}(n, t + 2k\pi)\,dt$$

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* Loc. cit. The analogous theorem referred to is Smith's Theorem 1.

† See, however, the footnote in which Smith's paper is first cited.

‡ We assume throughout this proof and the remainder of this section that the reader is thoroughly acquainted with Smith's paper, loc. cit. As an immediate consequence of Smith's Lemma 1, his Lemma 2 holds for all $t > 0$.

$$= \frac{1}{2}\sum_{k=-\infty}^{\infty}\left\{\sum_{j=1}^{r-1-2i}(-1)^{j-1}\cdot\frac{(r-1-2i)!}{(r-2i-j)!}t^{r-2i-j}\bar{\lambda}_{r+1,\beta}^{(r-j)}(n,t+2k\pi)\right]_{-\pi}^{\pi} + (-1)^{r-1-2i}(r-1-2i)!\int_{-\pi}^{\pi}\bar{\lambda}_{r+1,\beta}^{(2i+1)}(n,t+2k\pi)dt\Bigg\}$$

$$= \frac{1}{2}\sum_{j=1}^{r-1-2i}(-1)^{j-1}\frac{(r-1-2i)!}{(r-2i-j)!}\pi^{r-2i-j}\cdot\sum_{k=-\infty}^{\infty}\left\{\bar{\lambda}_{r+1,\beta}^{(r-j)}\{n,(2k+1)\pi\}-(-1)^{r-j}\bar{\lambda}_{r+1,\beta}^{(r-j)}\{n,(2k-1)\pi\}\right\}$$

$$= \sum_{j=i}^{[r/2]-1}(-1)^{r}\frac{(r-1-2i)!}{(r-2i-j)!}\pi^{2j-2i+1}\sum_{k=-\infty}^{\infty}\left\{\frac{-(2j+1)!}{\pi\{(2k+1)\pi\}^{2j+2}} + \frac{o(1)\text{ as }n\to\infty}{\{(2k+1)\pi\}^{2j+3}}\right\}$$

$$= \frac{(-1)^{r}}{2}\sum_{j=i}^{[r/2]-1}\frac{(r-1-2i)!}{(r-2i-j)!}\pi^{2j-2i}\frac{d^{2j+1}\operatorname{ctn}(s/2)}{ds^{2j+1}}\Bigg]_{s=\pi} + o(1)\text{ as }n\to\infty.$$

When r is even $[(r-1)/2]=[r/2]-1$, and when r is odd $[(r-1)/2]=[r/2]$, but the term where $i=[(r-1)/2]$ vanishes. The lemma follows when we multiply by the factor $2(-1)^{r}f^{(r-1-2i)}(x)/(r-1-2i)!$ and sum with respect to i.

PROOF OF THEOREM 2. To prove our Theorem 2 we replace $\omega_r(t)$ by $A_r(t)$ in the preliminary material and proof of Smith's Theorem 1; then using Lemma 2 we see that the nth partial Bosanquet-Linfoot (r,β) sum of the rth derived conjugate Fourier series is

$$(9)\quad J \equiv (-1)^{r+1}\int_{0}^{\infty}A_r(t)\bar{\lambda}_{r+1,\beta}^{(r)}(n,t)dt - C_r + o(1)\text{ as }n\to\infty.$$

Equation (9) takes the place of Smith's equation (13). To the right-hand sides of his equations (14) and (15) must be added, respectively, $C_{2r}+o(1)$ and $-C_{2r+1}+o(1)$. When modified as indicated above, Smith's proof establishes our theorem.

THE UNIVERSITY OF OREGON

ON SYMMETRIC BERNOULLI CONVOLUTIONS

BY AUREL WINTNER

In a previous note* a class of sequences $\{a_n\}$ has been determined such that

$$\prod_{n=1}^{\infty} \cos(a_n t) = O(\exp(-At^{\gamma})) \text{ as } t \to +\infty, \tag{1}$$

$$a_n > 0, \sum_{n=1}^{\infty} a_n^2 < +\infty,$$

where $A>0$ and $\gamma>0$. The role of this estimate was that of assuring a high degree of smoothness for the distribution function having the infinite product as Fourier-Stieltjes transform. In fact, this distribution function is an entire function if $\gamma>1$, it is regular analytic if $\gamma=1$ and it has derivatives of arbitrarily high order whenever there exists a $\gamma(>0)$. For the sequences $\{a_n\}$ in question and also for some more general sequences a simple proof of (1) will be given by means of an appraisal indicated in the reference cited. This appraisal has the advantage of being valid for *every* $\{a_n\}$ and it gives results also if (1) is not satisfied. It implies for instance the fact that the infinite product $\to 0$ as $t \to +\infty$ in cases of the type $a_n = c^{-n^{\delta}}$, where $c>1$ and $0<\delta<1$.

The nth factor of (1) does not come near to ± 1 if $1 \leqq a_n t \leqq 2$, since this limitation implies $|\cos(a_n t)| < 2/3$. Thus

$$\left| \prod_{n=1}^{\infty} \cos(a_n t) \right| < (2/3)^{K(t)},$$

if $1 \leqq a_n t \leqq 2$ is satisfied for $K = K(t)$ values of n. Now

$$K(t) \geqq N(2t) - N(t),$$

if $a_n \geqq 1/t$ holds for exactly $N(t)$ values of n.

Thus if $a_n = n^{-\alpha}$, then $N(t) = [t^{1/\alpha}]$, hence $K(t) > Ct^{1/\alpha}$ for

* A. Wintner, *On analytic convolutions of Bernoulli distributions*, American Journal of Mathematics, vol. 56 (1934), pp. 659–663.

some $C>0$ so that (1) is satisfied by $\gamma=1/\alpha$. Similarly, if $N(t)\sim t^{\beta}S(t)^{\pm 1}$, where $\beta>0$ and $S(t)$ is, as in the reference cited, a "slow" function, then (1) holds for every $\gamma<\beta$. If $1/a_n$ is the nth prime number, then (1) holds for every $\gamma<1$ since $K(t)>Ct^{\gamma}$ in virtue of the elementary inequalities of Tchebycheff. Thus it is not necessary to use the prime number theorem $\beta=1$, $S(t)^{\pm 1}=1/\log t$, applied in the reference cited. Correspondingly, the present method enables us to prove (1) also for sequences for which $N(t)$ is not $\sim t^{\beta}S(t)^{\pm 1}$.

THE JOHNS HOPKINS UNIVERSITY

PROOF OF THREE PROPOSITIONS OF PALEY

BY M. FEKETE

1. *Introduction.* In a letter to Fejér,* Paley stated three interesting propositions concerning Fourier series of bounded and continuous functions whose Fourier coefficients satisfy the conditions na_n, $nb_n \geqq -K$, $K \geqq 0$. The letter of Paley contains only a very brief sketch of the proof. After knowing Paley's results the author succeeded in developing complete proofs with various improvements of the estimates, and even in extending them to a wider class of Fourier series of almost periodic functions. These extensions will be treated elsewhere. In the present note we prove the following three theorems.

It will be assumed throughout that $f(x)$ is real-valued and periodic, of period 2π, and Lebesgue integrable over $(-\pi, \pi)$. Let

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \tag{1}$$

be its Fourier series expansion and let

$$s_n = s_n(x) = a_0 + \sum_{\nu=1}^{n} (a_\nu \cos \nu x + b_\nu \sin \nu x) \tag{2}$$

* This letter is reproduced in a note by Fejér, *On a theorem of Paley*, this Bulletin, vol. 40 (1934), pp. 469–475, especially pp. 474–475. It was communicated to the author by Professor Fejér in September, 1933.

be its partial sums. We also assume that the Fourier coefficients a_n, b_n satisfy the inequalities

$$na_n \geqq -K, \qquad nb_n \geqq -K, \tag{3}$$

where K is a constant that is not negative.

THEOREM A. *If $f(x)$ is bounded and*

$$\sup_x |f(x)| = L, \tag{4}$$

then

$$|s_n(x)| \leqq 5L + 6K. \tag{5}$$

THEOREM B. *Under the same conditions we have*

$$|s_n(x)| \leqq 5L + 8(KL)^{1/2}. \tag{6}$$

THEOREM C. *If $f(x)$ is continuous and conditions (3) are satisfied, then its Fourier series (1) converges to $f(x)$, uniformly for all x.**

2. *Proof of Theorem* A. The basis of the proof is the classical Fejér inequality, $|\sigma_n(x)| \leqq L$, for the nth arithmetic means of the sequence $\{s_n(x)\}$, which being written explicitly reduces to

$$\left| a_0 + \sum_{\nu=1}^{n} \left(1 - \frac{\nu}{n}\right)(a_\nu \cos \nu x + b_\nu \sin \nu x) \right| \leqq L, \quad (n \geqq 1). \tag{7}$$

Using the same idea as in the note of Fejér referred to above, we apply (7) to the Fourier series of $2\phi(x) \cos nx$ and $2\psi(x) \sin nx$ where

$$\phi(x) = \frac{1}{2}\{f(x) + f(-x)\} \sim a_0 + \sum_{\nu=1}^{\infty} a_\nu \cos \nu x, \tag{8}$$

$$\psi(x) = \frac{1}{2}\{f(x) - f(-x)\} \sim \sum_{\nu=1}^{\infty} b_\nu \sin \nu x. \tag{9}$$

* Theorems A and C incorporate the two theorems in Paley's letter, while Theorem B corresponds to his lemma. Theorems A and C were discovered independently, and proved by a different method by Szász, *Zur Konvergenztheorie der Fourierschen Reihen*, Acta Mathematica, vol. 61 (1933), pp. 185–201. Our estimate (5) is sharper than that of Szász, and there is no doubt that it is capable of further improvement. The interest of Theorems A and B lies in the fact that they furnish universal estimates for partial sums $s_n(x)$ of all functions of the class under consideration, in terms merely of L and K.

For every $n \geqq 1$ we have

$$(10)\quad 2\phi(x)\cos nx \sim \sum_{\nu=0}^{\infty} a_\nu \cos(\nu - n)x + \sum_{\nu=0}^{\infty} a_\nu \cos(\nu + n)x,$$

$$(11)\quad 2\psi(x)\sin nx \sim \sum_{\nu=1}^{\infty} b_\nu \cos(\nu - n)x - \sum_{\nu=1}^{\infty} b_\nu \cos(\nu + n)x.$$

On applying (7) we derive

$$(12)\quad \left| \sum_{\nu=0}^{n} \left(1 - \frac{n-\nu}{n}\right) a_\nu \cos(n-\nu)x + \sum_{\nu=n+1}^{2n} \left(1 - \frac{\nu - n}{n}\right) a_\nu \cos(\nu - n)x \right| \leqq 2L,$$

$$(13)\quad \left| \sum_{\nu=1}^{n} \left(1 - \frac{n-\nu}{n}\right) b_\nu \cos(n-\nu)x + \sum_{\nu=n+1}^{2n} \left(1 - \frac{\nu - n}{n}\right) b_\nu \cos(\nu - n)x \right| \leqq 2L,$$

whence, for $x = 0$,

$$(14)\quad \left| \sum_{\nu=1}^{n} \nu a_\nu + \sum_{\nu=n+1}^{2n} (2n - \nu) a_\nu \right| \leqq 2nL,$$

$$(15)\quad \left| \sum_{\nu=1}^{n} \nu b_\nu + \sum_{\nu=n+1}^{2n} (2n - \nu) b_\nu \right| \leqq 2nL.$$

Formula (14) being combined with (3), yields

$$0 \leqq \sum_{\nu=1}^{n} (\nu a_\nu + K) + \sum_{\nu=n+1}^{2n} \frac{2n-\nu}{\nu} (\nu a_\nu + K) \leqq 2n(L + K).$$

Now, again by (3),

$$(16)\quad \left| \sum_{\nu=1}^{n} \nu a_\nu \cos \nu x \right| \leqq \sum_{\nu=1}^{n} (\nu a_\nu + K) |\cos \nu x| + K \sum_{\nu=1}^{n} |\cos \nu x| \leqq (2L + 3K)n.$$

In similar fashion we derive

$$(17)\quad \left| \sum_{\nu=1}^{n} \nu b_\nu \sin \nu x \right| \leqq (2L + 3K)n.$$

On combining (7), (16), and (17), we immediately get (5).

3. *Proof of Theorem* B. We start with the identity indicated by Paley,

$$s_n(x) = \frac{n\sigma_n(x) - m\sigma_m(x)}{n-m} + R(x), \qquad (1 \leqq m < n), \tag{18}$$

where

$$\sigma_n(x) = a_0 + \sum_{\nu=1}^{n}\left(1 - \frac{\nu}{n}\right)(a_\nu \cos \nu x + b_\nu \sin \nu x), \tag{19}$$

$$R(x) = (n-m)^{-1} \sum_{\nu=m+1}^{n} (\nu - m)(a_\nu \cos \nu x + b_\nu \sin \nu x). \tag{20}$$

By (7)

$$\frac{|\, n\sigma_n(x) - m\sigma_m(x)\,|}{n-m} \leqq \frac{(n+m)L}{n-m}. \tag{21}$$

On applying (7) to the $(n-m)$th arithmetic means of the series (10) and (11) we have

$$\left| \sum_{0 \leqq n-\nu \leqq n-m} \left(1 - \frac{n-\nu}{n-m}\right) a_\nu \cos(n-\nu)x + \sum_{0 < \nu-n \leqq n-m} \left(1 - \frac{\nu-n}{n-m}\right) a_\nu \cos(\nu-n)x \right| \leqq 2L, \tag{22}$$

$$\left| \sum_{0 \leqq n-\nu \leqq n-m} \left(1 - \frac{n-\nu}{n-m}\right) b_\nu \cos(n-\nu)x + \sum_{0 \leqq \nu-n \leqq n-m} \left(1 - \frac{\nu-n}{n-m}\right) b_\nu \cos(\nu-n)x \right| \leqq 2L, \tag{23}$$

whence, for $x = 0$,

$$\left| \sum_{\nu=m+1}^{n} (\nu - m)a_\nu + \sum_{\nu=n+1}^{2n-m} (2n - m - \nu)a_\nu \right| \leqq 2L(n-m), \tag{24}$$

$$\left| \sum_{\nu=m+1}^{n} (\nu - m)b_\nu + \sum_{\nu=n+1}^{2n-m} (2n - m - \nu)b_\nu \right| \leqq 2L(n-m). \tag{25}$$

Formula (24), in view of (3), yields

$$0 \leqq \sum_{\nu=m+1}^{n} (\nu - m)(a_\nu + K/\nu) \leqq \sum_{\nu=m+1}^{n} (\nu - m)(a_\nu + K/\nu)$$
$$+ \sum_{\nu=n+1}^{2n-m} (2n - m - \nu)(a_\nu + K/\nu) \leqq (n - m)\left(2L + K \sum_{\nu=m+1}^{2n-m} 1/\nu\right).$$

Thus we have

$$\left| \sum_{\nu=m+1}^{n} (\nu - m) a_\nu \cos \nu x \right| \leqq \left| \sum_{\nu=m+1}^{n} (\nu - m)(a_\nu + K/\nu) \cos \nu x \right|$$
$$+ \left| \sum_{\nu=m+1}^{n} (\nu - m)(K/\nu) \cos \nu x \right| \tag{26}$$
$$\leqq \sum_{\nu=m+1}^{n} (\nu - m)(a_\nu + K/\nu) + K \sum_{\nu=m+1}^{n} (\nu - m)/\nu$$
$$\leqq (n - m)\left(2L + K \sum_{\nu=m+1}^{2n-m} 1/\nu + K \sum_{\nu=m+1}^{n} 1/\nu\right)$$

and, similarly,

$$\left| \sum_{\nu=m+1}^{n} (\nu - m) b_\nu \sin \nu x \right| \tag{27}$$
$$\leqq (n - m)\left(2L + K \sum_{\nu=m+1}^{2n-m} 1/\nu + K \sum_{\nu=m+1}^{n} 1/\nu\right).$$

Taking into account the well known inequality

$$\sum_{\nu=\alpha+1}^{\beta} 1/\nu < \log \frac{\beta}{\alpha}, \qquad (1 \leqq \alpha < \beta),$$

we obtain an estimate for $R(x)$,

$$|R(x)| \leqq 4L + 2K \log (n(2n - m)m^{-2}), \quad (1 \leqq m < n). \tag{28}$$

From (18), (21), (28) it results that, for $1 \leqq m < n$,

$$|s_n(x)| \leqq L\left(4 + \frac{n + m}{n - m}\right) + 2K \log \frac{n(2n - m)}{m^2}. \tag{29}$$

We propose to show that (29) and (5) imply, for an arbitrary $\delta > 0$,

$$|s_n(x)| \leqq 5L + 4(L/\delta + K\delta). \tag{30}$$

If this is shown to be true, then (6) of Theorem B follows at once, since always

$$(KL)^{1/2} \leqq \frac{1}{2}(L/\delta + K\delta),$$

with the equality sign (only) in case $\delta = (L/K)^{1/2}$ and $K > 0$.

To derive (30) from (29), we observe that, unless

$$n < \frac{1}{2}(2+\delta)(1+e^{\delta})(e^{\delta}-1-\delta)^{-1}, \tag{31}$$

the interval $(2n(1+e^{\delta})^{-1},\ 2n(2+\delta)^{-1})$ contains at least one integer m. For such an m we have

$$1 + \frac{\delta}{2} < \frac{n}{m} < \frac{1}{2}(1+e^{\delta}) < e^{\delta},$$

$$\frac{n+m}{n-m} = 1 + 2(n/m-1)^{-1} < 1 + \frac{2}{\delta/2} = 1 + \frac{4}{\delta},$$

$$\log\,(n(2n-m)m^{-2}) = \log\left(\frac{n}{m}\right) + \log\left(\frac{2n}{m} - 1\right) < \log e^{\delta} + \log e^{\delta} = 2\delta,$$

whence, by (29),

$$|\,s_n(x)\,| \leqq 5L + 4(L/\delta + K\delta).$$

Suppose now that (31) holds; (30) follows immediately from (5) if $\delta \geqq 3/2$. Finally, if $0 < \delta < 3/2$, then, by the Cauchy-Schwarz inequality and Parseval's theorem,

$$\begin{aligned} |\,s_n(x)\,| &\leqq |\,a_0\,| + \sum_{\nu=1}^{n} |\,a_\nu \cos \nu x + b_\nu \sin \nu x\,| \\ &\leqq |\,a_0\,| + \left(n \sum_{\nu=1}^{n} (a_\nu^2 + b_\nu^2)\right)^{1/2} \leqq (1 + (2n)^{1/2})L \\ &\leqq \{1 + (2+\delta)^{1/2}(1+\delta)^{1/2}(e^{\delta}-1-\delta)^{-1/2}\}L. \end{aligned}$$

But, if $0 < \delta < 3/2$, we have

$$\begin{aligned} e^{\delta} &< 3 + 4\delta, \\ 16(1+\delta)^2 > 4(1+\delta)(1+e^{\delta}) &> 2\delta^2(1+\delta)(1+e^{\delta})(e^{\delta}-1-\delta)^{-1} \\ &> \delta^2(2+\delta)(1+e^{\delta})(e^{\delta}-1-\delta)^{-1}, \\ 1 + (2+\delta)^{1/2}(1+e^{\delta})^{1/2}(e^{\delta}-1-\delta)^{-1/2} &< 5 + \frac{4}{\delta}, \end{aligned}$$

and again (30) follows. Thus (30) holds for all $\delta>0$ and the proof of Theorem B is now complete*.

4. *Proof of Theorem* C. Theorem C is now easily derived from Theorem B. Since $f(x)$ is uniformly continuous we have by Fejér's theorem,

$$\max_x \left| f(x) - \sigma_n(x) \right| = M_n \to 0 \text{ as } n \to \infty . \tag{32}$$

The Fourier coefficients of the function $f(x)-\sigma_n(x)$ are given by

$$\alpha_\nu^{(n)} = \frac{\nu a_\nu}{n}, \qquad \beta_\nu^{(n)} = \frac{\nu b_\nu}{n}, \qquad (1 \leqq \nu \leqq n-1),$$

$$\alpha_\nu^{(n)} = a_\nu, \qquad \beta_\nu^{(n)} = b_\nu, \qquad (\nu \geqq n).$$

Conditions (3) obviously imply analogous conditions

$$\nu\alpha_\nu^{(n)} \geqq -K, \qquad \nu\beta_\nu^{(n)} \geqq -K.$$

On applying Theorem B to the Fourier series of $f(x)-\sigma_n(x)$ we have

$$\begin{aligned} \left| s_n(x) - \sigma_n(x) \right| &= \left| \sum_{\nu=1}^{n} \left(\frac{\nu}{n}\right) (a_\nu \cos \nu x + b_\nu \sin \nu x) \right| \\ &\leqq 5M_n + 8(KM_n)^{1/2}, \end{aligned} \tag{33}$$

and finally,

$$\begin{aligned} \left| f(x) - s_n(x) \right| &\leqq \left| f(x) - \sigma_n(x) \right| + \left| \sigma_n(x) - s_n(x) \right| \\ &\leqq 6M_n + 8(KM_n)^{1/2}. \end{aligned} \tag{34}$$

This proves Theorem C and, at the same time, gives information concerning the rapidity with which $s_n(x)$ tends to its limiting function $f(x)$.

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* Formula (30) is of the type given by Paley, $|s_n(x)| \leqq K\delta + LM_\delta$ (loc. cit., p. 474), but has the advantage of giving a simple expression for M_δ.

OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

NEW YORK CITY, February 23, 1935.

Abstracts must be in the hands of Associate Secretary J. R. Kline, 501 West 116th St., New York City, not later than January 26. Abstracts received by the Secretaries on or before January 11 will appear in the January issue of this Bulletin. By invitation of the Program Committee, Professor Morgan Ward will deliver an address on the topic *Arithmetic theory of linear recurring series.*

STANFORD UNIVERSITY, April 6, 1935.

Abstracts must be in the hands of Associate Secretary T. M. Putnam, University of California, Berkeley, California, not later than March 9. All these will appear in the March issue of this Bulletin.

NEW YORK CITY, April 19-20, 1935.

Abstracts must be in the hands of Associate Secretary J. R. Kline, 501 West 116th St., New York City, not later than March 23. Abstracts received by the Secretaries on or before March 9 will appear in the March issue of this Bulletin. There will be a symposium on the topic *The problem of three bodies, and allied problems,* at which the speakers will be Professors G. D. Birkhoff, E. W. Brown, and Aurel Wintner.

CHICAGO, ILLINOIS, April 19-20, 1935.

Abstracts must be in the hands of Associate Secretary M. H. Ingraham, University of Wisconsin, Madison, Wisconsin, not later than March 23. Abstracts received by the Secretaries on or before March 9 will appear in the March issue of this Bulletin. By invitation of the Program Committee, Professor L. M. Graves will deliver an address on *Topics in Functional Calculus.*

R. G. D. RICHARDSON, *Secretary of the Society.*

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